LOCAL TIMES AND TANAKA–MEYER FORMULAE FOR CÂDLÀG PATHS

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ABSTRACT. Three concepts of local times for deterministic càdlàg paths are developed and the corresponding pathwise Tanaka–Meyer formulae are provided. For semimartingales, it is shown that their sample paths a.s. satisfy all three pathwise definitions of local times and that all coincide with the classical semimartingale local time. In particular, this demonstrates that each definition constitutes a leg pathwise counterpart of probabilistic local times.

Keywords: càdlàg path, Föllmer–Itô formula, local time, pathwise stochastic integration, pathwise Tanaka formula, semimartingale.

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1. Introduction

Stochastic calculus, with its foundational notions developed by Kyōsō Itō in the 1940s, is a par excellence probabilistic endeavour. The stochastic integral, the integration by parts formula – these basic building blocks are to be understood almost surely, and so is the edifice they span. This thinking has proved to be exceedingly powerful and fruitful, and underpins many a beautiful development in probability theory since then. Nevertheless, for decades now, mathematicians have been trying to develop a more analytic, pathwise understanding of these probabilistic objects. On one hand, this was, and is, driven by mathematical curiosity. The classical calculus remains an irresistible reference point and, e.g., in developing a notion of an integral it is important to understand when and how it can be seen as a limit of its Riemann sums. On the other hand, this was, and is, driven by applications. Stochastic differential equations have become a ubiquitous tool for mathematical modelling from physics, through biology to finance. Yet, they do not offer the same level of path-by-path description of the system’s evolution as the classical differential equations do. This becomes particularly problematic if need to work simultaneously with many probability measures, possibly mutually singular, as happens when dealing with model uncertainty. One field where this is important, and which has driven renewed interest in pathwise stochastic calculus, is robust mathematical finance, see for example [DOR14] and the references therein. Both of the above reasons - mathematical curiosity and possible applications - are important for us. We add to this literature and develop a pathwise approach to stochastic calculus for càdlàg paths using local times.

In his seminal paper [Föl81], H. Föllmer introduced, for twice continuously differentiable \( f: \mathbb{R} \to \mathbb{R} \), a non-probabilistic version of the Itô formula

\[
 f(x_t) - f(x_0) = \int_0^t f'(x_s) \, dx_s + \frac{1}{2} \int_0^t f''(x_s) \, \text{d}[x]_s + J_f(x), \quad t \in [0, T],
\]

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where \( x: [0, T] \rightarrow \mathbb{R} \) is càdlàg and possesses a suitably defined quadratic variation \([x]\) such that
\[
[x] = [x]^c + \sum_{0 < s \leq t} (\Delta x_s)^2,
\]
where \( \Delta x_t := x_{t-} - x_t \), and \( J^f_t(x) \) is defined by the following absolutely convergent series
\[
J^f_t(x) := \sum_{0 < s \leq t} (\Delta f(x_s) - f'(x_{s-}) \Delta x_s).
\]
In particular, this leads to a pathwise definition of the “stochastic” integral \( \int_0^t f'(x_s- ) \, dx_s \), assuming \([x]\) exists. Soon after, Stricker \cite{Str81}, showed that one could not extend the above to all continuous functions \( f \). This could only be done adopting a much more bespoke discretisation and probabilistic methods, see for example \cite{Bic81, Kar95}. Accordingly, the main remaining challenge was to understand the case of functions \( f \) which are not twice continuously differentiable but are weakly differentiable, in some sense. Classically, this realm is covered by the Tanaka–Meyer formula.

For continuous paths Föllmer’s pathwise Itô formula was generalized to a pathwise Tanaka–Meyer formulae in the early works of \cite{Wue80, Lem83} and more recently in \cite{PP15} and \cite{DOS18}, who offered a comprehensive study. Furthermore, we refer to \cite{GH80} and \cite{BY14, DOR14} for related work in a pathwise spirit. Our contribution here is to study this problem for càdlàg paths. Jump processes, e.g., Lévy processes, are of both theoretical and practical importance and, as above, our study is motivated by both mathematical curiosity as well as applications. Already in the classical, probabilistic, setting stochastic calculus for jump processed requires novel insights over and above the continuous case. This was also observed in recent works focusing on Föllmer’s Itô calculus for càdlàg paths, see \cite{CC18} and \cite{Hir19}. We face the same difficulty, which of course makes our study all the more interesting. In particular, we need more information and new ideas to handle jumps. This is consistent with the definition of quadratic variation for càdlàg paths, cf. \cite{CC18}.

Our non-probabilistic versions of Tanaka–Meyer formula, extend the above Itô formula allowing for functions \( f \) with weaker regularity assumptions than \( C^2 \). More precisely, we derive pathwise formulae
\[
f(x_t) - f(x_0) = \int_{[0,t]} f'(x_s-) \, dx_s + \frac{1}{2} \int_{\mathbb{R}} L_t(x,u) f''(du) + J^f_t(x), \quad t \in [0,T],
\]
for twice weakly differentiable functions \( f \), supposing that the càdlàg path \( x \) possesses a suitable pathwise local time \( L(x) \). As in the case of the Itô formula, there exists no unique pathwise sense to understand such a formula, see also Remark 2.15 below. We develop three natural pathwise approaches to local times and, consequently, to their stochastic calculus. First, we start with the key property relating local times and quadratic variation: the time-space occupation formula, and use it define pathwise local times. Second, in the spirit of \cite{Fol81, Wue80}, we discretise the path along a sequence of partitions and obtain local times as limits of discrete level crossings and stochastic integrals as limits of their Riemann sums. Finally, we discretise the integrand via the Skorokhod map which provides a natural approximation of the “stochastic” integral and links to the concept of truncated variation. In all of the three cases we show that a pathwise variant of the Tanaka–Meyer formula holds. Further, we prove that for a càdlàg semimartingale, all three constructions coincide a.s. with classical local times. This shows that all three approaches are legitimate extensions of the classical stochastic results to pathwise analysis. Each has its merits and limitations which we explore.
in detail. Our aim is to provide a comprehensive understanding of how to deal with jumps in the context of pathwise Tanaka–Meyer formula. We thus do not seek further extensions of the setup, e.g., to cover time-dependent functions \( f \), cf. [FZ06], path-dependent functions, cf. [CF10; IP15; Sap18], nor to develop higher order local time in the spirit of [CP19] for càdlàg paths. These, while interesting, would distract from the main focus of the paper and are left as avenues for future research.

**Outline:** In Section 2 we propose three notions of local times for càdlàg paths and establish the corresponding Tanaka–Meyer formulae. Then, in Section 3 we show that sample paths of semimartingales almost surely possess such local times and all three definitions agree a.s. in the classical stochastic world.

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2. **Pathwise local times and Tanaka–Meyer formulae**

The first non-probabilistic version of Itô’s formula and the corresponding notion of pathwise quadratic variation of càdlàg paths was introduced by H. Föllmer in the seminal paper [Fol81]. Before providing non-probabilistic versions of Tanaka–Meyer formulae and introducing the corresponding pathwise local times, we recall in the next subsection some results from [Fol81].

2.1. **Quadratic variation and Föllmer–Itô formula.** For \( T \in (0, \infty) \), let \( D([0, T]; \mathbb{R}) \) be the space of all càdlàg (RCLL) functions \( x: [0, T] \to \mathbb{R} \), that is, \( x \) is right-continuous and possesses left-limits at each \( t \in [0, T] \). For \( x \in D([0, T]; \mathbb{R}) \) we set \( x_{t-} := \lim_{s \downarrow t} x_s \) for \( t \in (0, T) \), \( x_0 := x_0 \) and \( \Delta x_s := x_s - x_{s-} \) for \( s \in [0, T] \). In order to define the summation over the jumps of a càdlàg function, we need the concept of summation over general sets, see for example [Kel75] p.77-78. Let \( I \) be a set, let \( b: I \to \mathbb{R} \) be a real valued function and let \( I \) be the family of all finite subsets of \( I \). Since \( I \) is directed when endowed with the order of inclusion \( \subseteq \), the summation over \( I \) can be defined by

\[
\sum_{i \in I} b_i := \lim_{J \to I} \sum_{i \in J} b_i
\]

as limit of a net, i.e., \( \lim_{J \in I} \sum_{i \in J} b_i =: l \in \mathbb{R} \) exists if for all \( \delta > 0 \) there is a \( \varepsilon > 0 \) and \( J \in I \) such that for all \( K \in I \) such that \( K \supseteq J \) (i.e. \( K \supseteq J \)) one has \( |l - \sum_{i \in K} b_i| < \varepsilon \).

For a continuous function \( f: \mathbb{R} \to \mathbb{R} \) possessing a left-derivative \( f' \), we set

\[
J^f_t(x) := \sum_{0 < s \leq t} (\Delta f(x_s) - f'(x_{s-}) \Delta x_s),
\]

provided the sum exists. Furthermore, the space of continuous functions \( f: \mathbb{R} \to \mathbb{R} \) is denoted by \( C(\mathbb{R}) := C(\mathbb{R}; \mathbb{R}) \), the space of twice continuously differentiable functions by \( C^2(\mathbb{R}) := C^2(\mathbb{R}; \mathbb{R}) \) and the space of smooth functions by \( C^\infty(\mathbb{R}) := C^\infty(\mathbb{R}; \mathbb{R}) \).

A **partition** \( \pi = (t_j)_{j=0}^N \) is a finite sequence such that \( 0 = t_0 < t_1 < \cdots < t_N = T \) (for some \( N \in \mathbb{N} \)). We write \( |\pi| := \max_{j \in \mathbb{N}} |t_j - t_{j-1}| \) for its mesh size and define \( \pi(t) := \pi \cap [0, t] \) the restriction of \( \pi \) to \([0, t]\). A sequence of partitions \( (\pi^n)_{n \in \mathbb{N}} \) is said to be **refining** if for all \( t_j \in \pi^n \) we also have \( t_j \in \pi^{n+1} \) and a refining sequence \( (\pi^n)_{n \in \mathbb{N}} \) is said to exhaust the jumps of \( x \) if for all \( t \in [0, T] \) with \( \Delta x_t \neq 0 \), \( t \in \pi^n \) for \( n \) large enough. The Dirac measure at \( t \in [0, T] \) is denoted by \( \delta_t \).
**Definition 2.1.** Let \((\pi^n)_n\) be a sequence of partitions such that \(\lim_{n \to \infty} |\pi^n| = 0\). A function \(x \in D([0,T]; \mathbb{R})\) has quadratic variation along \((\pi^n)_n\) if the sequence of discrete measures

\[
\mu_n := \sum_{t_j \in \pi^n} (x_{t_{j+1}} - x_{t_j})^2 \delta_{t_j}
\]

converges vaguely to a Radon measure \(\mu\) with \([x]_t := \mu((0,t])\), such that

\[
[x]^c_t := [x]_t - \sum_{0 < s < t} (\Delta x_s)^2
\]

is an increasing continuous function.

\(Q((\pi^n)_n)\) denotes the set of functions in \([0,T]; \mathbb{R}\) having quadratic variation along \((\pi^n)_n\).

Following the convention that \(\int_0^t\) stands for \(\int_{(0,t]}\), H. Föllmer provided a pathwise version of Itô’s formula with respect to paths in \(Q((\pi^n)_n)\).

**Theorem 2.2 ([Föll81]).** Let \(x \in Q((\pi^n)_n)\) and \(f \in C^2(\mathbb{R})\). Then, the pathwise Itô formula

\[
f(x_t) - f(x_0) = \int_0^t f'(x_{s-}) \, dx_s + \frac{1}{2} \int_0^t f''(x_s) \, [x]^c_s + J'_t(x), \quad t \in [0,T],
\]

holds with \(J'_t(x)\) as in (2.2), and with

\[
\int_0^t f'(x_{s-}) \, dx_s := \lim_{n \to \infty} \sum_{t_j \in \pi^n(t)} f'(x_{t_j})(x_{t_{j+1}} - x_{t_j}), \quad t \in [0,T],
\]

where the series in (2.2) is absolutely convergent and the limit in (2.4) exists.

In our proofs, we will use Föllmer’s pathwise Itô formula [2.3]. We remark once and for all that, while to define \(\int_0^t f'(x_{s-}) \, dx_s\) Föllmer [Föll81] takes limits of sums of the form

\[
\sum_{\pi_n \ni t_j \leq t} g(x_{t_j})(x_{t_{j+1}} - x_{t_j}), \quad \text{whereas we consider} \quad \sum_{t_j \in \pi_n} g(x_{t_j})(x_{t_{j+1} \wedge t} - x_{t_j \wedge t}),
\]

this has no consequences, since the difference between these two sums is

\[
g(x_{t_{c(\pi,t)}})(x_{t_{c(\pi,t)}} - x_t) \quad (\text{where} \ c(\pi,t) := \max\{j : \pi \ni t_j \leq t\}),
\]

which goes to zero as \(|\pi| \to 0\) since \(g\) is bounded on \([\inf_{t \in [0,T]} x_t, \sup_{t \in [0,T]} x_t]\), \(x\) is càdlàg and \(t < t_{c(\pi,t)+1} \leq t + |\pi|\). So Föllmer’s pathwise Itô formula [2.3] does indeed hold also with our definition of \(\int_0^t f'(x_{s-}) \, dx_s\). Notice that analogously

\[
\sum_{\pi_n \ni t_j \leq t} g(x_{t_j})(x_{t_{j+1}} - x_{t_j})^2 \quad \text{and} \quad \sum_{t_j \in \pi_n} g(x_{t_j})(x_{t_{j+1} \wedge t} - x_{t_j \wedge t})^2
\]

differ by

\[
g(x_{t_{c(\pi,t)}})((x_{t_{c(\pi,t)}} - x_t)^2 - (x_t - x_{t_c})^2), \quad \text{where} \ c := c(\pi_n,t),
\]

which goes to zero as \(|\pi_n| \to 0\).
2.2. Local time via occupation measure. In order to extend the Itô formula for twice continuously differentiable functions $f$ to twice weakly differentiable functions $f$, the notion of quadratic variation is not sufficient and the concept of local time is required. In probability theory there exist various classical approaches to define local times of stochastic processes. In the present deterministic setting, we first introduce a pathwise local time corresponding to the notion of local time as occupation measure with respect to the quadratic variation.

The space of $q$-integrable functions $g: \mathbb{R} \to \mathbb{R}$ is denoted by $L^q(\mathbb{R}) := L^q(\mathbb{R}; \mathbb{R})$ with corresponding norm $\| \cdot \|_{L^q}$ for $q \in [1, \infty]$ and $W^{k,q}(\mathbb{R}) := W^{k,q}(\mathbb{R}; \mathbb{R})$ stands for the Sobolev space of functions $g: \mathbb{R} \to \mathbb{R}$ which are $k$-times weakly differentiable in $L^q(\mathbb{R})$, for $k \in \mathbb{N}$. Moreover, $L^q(K; \mathbb{R})$ is the space of $q$-integrable functions $f: K \to \mathbb{R}$ for a Borel set $K \subset \mathbb{R}$ and we recall the left-continuous sign-function

$$\text{sign}(x) := \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0 \end{cases}.$$

We define, for $a, b \in \mathbb{R}$,

$$\|a, b\| := \begin{cases} [a, b) & \text{if } a \leq b \\ [b, a) & \text{if } a > b \end{cases} \quad \text{with} \quad [a, a) := \emptyset.$$

**Definition 2.3.** Let $x \in \mathcal{Q}((\pi^n)_n)$. A Borel function $L.(x, \cdot): [0, T] \times \mathbb{R} \to [0, \infty)$ is called the occupation local time of $x$ if

$$\int_{-\infty}^{t} g(u) L_{t}(x, u) \, du = \int_{0}^{t} g(x_s) \, d[x]_s^{c}, \quad t \in [0, T],$$

holds for any positive Borel function $g: \mathbb{R} \to [0, \infty)$.

To extend Itô’s formula to a Tanaka–Meyer formula, as e.g. in [Pro04], we will consider the quantity $J_{t}(x, \cdot) := J^{fu}_{t}(x)$ where $f_u := |\cdot - u|/2$. We will at times drop $x$ from the notation, and simply write $L_{t}(u)$ and $J_{t}(u)$. It is straightforward to verify\(^1\) that

$$|x_u - u| - |x_{s-} - u| - \text{sign}(x_{s-} - u) \Delta x_s = 2|x_s - u|1_{[x_{s-}, x_s)},$$

which yields the useful compact expression

$$J_{t}(x, u) = \sum_{0 < s \leq t} |x_s - u|1_{[x_{s-}, x_s]}(u), \quad u \in \mathbb{R},$$

which also shows that $J$, like $L$, is a positive function; in particular, $L_{t}(\cdot)/2 + J_{t}(\cdot) \in L^{p}(\mathbb{R})$ if and only if $L_{t}(\cdot), J_{t}(\cdot) \in L^{p}(\mathbb{R})$. Notice, $x$ is bounded (since it is càdlàg), and that $L_{t}(u)$ and $J_{t}(u)$ equal 0 if $u$ does not belong to the compact set $[\inf_{s \in [0, T]} x_s, \sup_{s \in [0, T]} x_s]$.\(^2\)

**Definition 2.4.** The set $\mathcal{L}_{p}((\pi^n)_n)$ denotes all paths $x \in \mathcal{Q}((\pi^n)_n)$ having an occupation local time $L$ and such that $K_{t}(x, \cdot) := L_{t}(x, \cdot)/2 + J_{t}(x, \cdot) \in L^{p}(\mathbb{R})$ for all $t \in [0, T]$.

There is no common agreement in the related literature in probability theory as to whether $L$ or $L/2$ is to be called local time, cf. [KSSS, Remark 6.4]; here we decided to follow the convention made in the standard textbook [Pro04]. A classical approach to extend Itô’s formula and, in particular, the “stochastic” integral $\int_{0}^{t} f'(x_{s-}) \, dx_s$ to twice weakly differentiable functions $f$, is to approximate the function $f$ by smooth functions, cf. [KSSS, Theorem 3.6.22] with the function $f_u(\cdot) := |\cdot - u|/2$.

\(^1\)Either checking separately the six cases where $u \leq x_{s-} \leq x_s$, $x_{s-} \leq u \leq x_s$, etc., or using the identity (2.11)

\(^2\)with the function $f_u(\cdot) := |\cdot - u|/2$.\(^2\)
for the case of Brownian motion. For this purpose we consider a “mollifier” \( \rho \), i.e., a positive function \( \rho \in C^\infty(\mathbb{R}) \) and such that \( \int_{-\infty}^\infty \rho(u) \, du = 1 \), and set \( \rho_n(u) := n\rho(nu) \) for \( n \in \mathbb{N} \).

Given a function \( f \in W^{2,q}(\mathbb{R}) \) we approximate it via the convolution \( f_n := \rho_n * f \). In this way, \( f_n \in C^2(\mathbb{R}) \), \( f_n \to f \) in \( W^{2,q}(\mathbb{R}) \) if \( q < \infty \) (if \( q = \infty \) this is true if one assumes \( f'' \) is continuous) and, in particular, \( \lim_{n \to \infty} f_n(x) = f(x) \) for \( x \in \mathbb{R} \).

**Proposition 2.5.** Let \( x \in L_p((\pi^n)_n) \) and \( f \in W^{2,q}(\mathbb{R}) \) with \( 1/p + 1/q \geq 1 \) and \( q \in [1, \infty) \). Then, the series \( \sum_{n=1}^\infty f_n(x) \) defining \( J_f^L(x) \) is absolutely convergent, \( \int_0^t f_n(x_s) \, dx_s \) defined by (2.4) converges to the finite limit

\[
\int_0^t f'(x_s^-) \, dx_s := \lim_{n \to \infty} \int_0^t f_n(x_s^-) \, dx_s, \quad t \in [0,T], \tag{2.8}
\]

which does not depend on the choice of \( \rho \), and the pathwise Tanaka–Meyer formula

\[
f(x_t) - f(x_0) = \int_{[0,t]} f'(x_s^-) \, dx_s + \frac{1}{2} \int_{\mathbb{R}} L_t(x,u)f''(du) + J_f^L(x), \quad t \in [0,T], \tag{2.9}
\]

holds with such definition of \( \int_0^t f'(x_s^-) \, dx_s \). The statements hold for \( q = \infty \) if \( f'' \) is continuous.

Because of Proposition 2.5, it is of interest to ask under which assumptions one can get that \( L_t(x,\cdot) \) and \( J_f^L(x,\cdot) \) are in \( L^p(\mathbb{R}) \). First, remark that, since both quantities are equal to \( |x|^a \) and \( |x|^a \) is continuous, the \( p \)-integrability requirement in Definition 2.4 is a local one. Then, notice that if \( x \in Q((\pi^n)_n) \) has an occupation local time then \( L_t, J_f^L \in L^1(\mathbb{R}) \) (i.e. \( x \in L^1((\pi^n)_n) \)), since

\[
\int_{\mathbb{R}} L_t(x,u) \, du = [x]_t^c < \infty, \quad \int_{\mathbb{R}} J_f^L(x,u) \, du = \frac{1}{2} [x]_t^d < \infty,
\]

where \( [x]_t^d \) denotes the jump part of the increasing càdlàg function \( [x] \), i.e.,

\[
[x]_t^d := \sum_{0<s \leq t}(\Delta x_s)^2.
\]

**Remark 2.6.** If \( p \in [1, \infty) \) and \( C_p := 1/(p + 1)^{1/p} \) then

\[
\|J_f^L(x,\cdot)\|_{L^p} \leq C_p \sum_{0<s \leq t} |\Delta x_s|^{1+\frac{1}{p}}.
\]

This can be seen as a consequence of Minkowski’s integral inequality and of the identity

\[
\int_{[a,b]} |b-u|^p \, du = \frac{|b-a|^{p+1}}{p+1}.
\]

A similar bound for \( L_t \) can be given under the stronger assumption \( x \in L^{3W}_p((\pi^n)_n) \), see Definition 2.17 and equation (2.21) in the next subsection. An alternative criterion of \( p \)-summability for \( L_t \), for \( p \in (1, \infty) \), which clearly holds under the assumption \( x \in L_p((\pi^n)_n) \), is:

\[
\|L_t(x,\cdot)\|_{L^p} = \sup \left\{ \int_0^t g(x_s) \, dx_s^c : \|g\|_{L^q} \leq 1 \right\} < \infty.
\]

Notice that an occupation local time \( L_t \) is only unique up to equality a.e. \( u \) for each \( t \); in particular, \( L_t \) could be thought of as an equivalence class, and one is then lead to look for good representatives. In particular, it is often of interest to have a version \( L_t \) which is càdlàg in \( t \); here is one way to do it, using for \( L \) the fact that \( L_s \leq L_t \) a.e. if \( s \leq t \), \( L_t \in L^1(\mathbb{R}) \)
for all \( t, t' \mapsto \int_{\mathbb{R}} L_t(u) \, du = [x]^x_t\) is continuous and a standard\(^2\) regularization technique. One can also prove that \( J \) is càdlàg using the dominated convergence theorem, and the fact that \( J_t \in L^1 \) for all \( t \).

**Remark 2.7.** If \( x \) has an occupation local time \( L \), then one can choose for each \( t \in [0, T] \) a version \( L_t(\cdot) \) of \( L_t \) such that \( L(\cdot) \) is finite and càdlàg increasing for each \( u \in \mathbb{R} \). Moreover, \( J(u) \) is finite and càdlàg increasing for a.e. \( u \).

Remark 2.8. Notice that

\[
\text{var}(J_t(x, \cdot)) = \sup \left\{ \sum_{i=0}^{N-1} |J_t(x, u_{i+1}) - J_t(x, u_i)| : (u_i)_{i=0}^N \subset \mathbb{R}, N \in \mathbb{N} \right\} \leq \sum_{0<s \leq t} |\Delta x_s|,
\]

and so if \( \sum_{0<s \leq t} |\Delta x_s| < \infty \) for all \( t \), then \( J_t(x, \cdot) \) is càdlàg and of finite variation for all \( t \in [0, T] \).

As an application of having a version \( L \) of \( L \) which is càdlàg in \( t \), notice that the occupation time formula \((2.5)\) then extends to all positive Borel \( h = h(s, u) \) as follows

\[
\int_{-\infty}^{\infty} \left( \int_0^t h(\cdot, u) \, dL_x(x, u) \right) \, du = \int_0^t h(s, x_s) \, d[x]_s^c, \quad t \in [0, T].
\]

Moreover, if \( J \) is càdlàg in \( t \) it also satisfies a restricted occupation time formula: if \( h = h(s, u) \) is a positive Borel function such that \( h(s, u) = h(s, x_s) \) for a.e. \( u \in [x_{s-}, x_s] \), then Fubini’s theorem gives that

\[
\int_{-\infty}^{\infty} \left( \int_0^t h(\cdot, u) \, dJ_x(x, u) \right) \, du = \frac{1}{2} \int_0^t h(s, x_s) \, d[x]_s^d,
\]

and this observation seems to be new.

To prove Proposition 2.5 and for later use, let us recall some well known facts. Recall that \( \nu : \mathbb{R} \to \mathbb{R} \) is convex iff its second distributional derivative \( \nu'' \) is a positive Radon\(^3\) measure; thus \( f : \mathbb{R} \to \mathbb{R} \) equals the difference of two convex functions iff \( f'' \) is a (real) Radon measure; \( |f''| \) then denotes its total variation. Given such \( f, f' \) denotes the left-derivative of \( f \), which is left-continuous and of locally bounded variation and satisfies \( f(b) - f(a) = \int_a^b f'(y) \, dy \) for all \( a, b \in \mathbb{R} \). Thus for \( b \geq a \) we get that

\[
f(b) - f(a) - f'(a)(b-a) = \int_a^b (f'(u) - f'(a)) \, du = \int_{[a,b]} (b-u) \, f''(du),
\]

where we used integration by parts. For \( b < a \), we get instead

\[
f(b) - f(a) - f'(a)(b-a) = \int_{[b,a]} (u-b) \, f''(du),
\]

\(^2\)In a probabilistic setting one could also use that \(-L\) is a supermartingale with respect to the constant filtration \( \mathcal{F}_t := \mathcal{F} \) and apply [Pro03] Chapter 1, Theorem 9.

\(^3\)I.e. it is finite on every compact \( K \subseteq \mathbb{R} \).
so we obtain the identity

\[(2.11) \quad J^f(a, b) := f(a) - f(b) - f'(b)(a - b) = \int_{[a,b]} |b - u| f''(du), \quad a, b \in \mathbb{R},\]

which can often be used in proofs in lieu of the following representation

\[(2.12) \quad f(x) = ax + b + (||x||^p)'(x), \quad x \in \mathbb{R},\]

(which holds for some $a, b \in \mathbb{R}$), which is often used in the literature. Representation (2.12) holds whenever $\int_{\mathbb{R}} |a - u| f''(du) < \infty$ for all $a$ (in particular if $f''$ has compact support), and is proved after Proposition 3.2 in [RY99, Appendix 3].

A version of the following statement appears without proof during the course of the proof of [Pro04, Chapter 4, Theorem 70].

**Lemma 2.9.** Assume $f: \mathbb{R} \to \mathbb{R}$ equals the difference of two convex functions. Then, if $f$ is convex the series $f^J(x)$ consists only of positive terms, whereas if

\[ \int_{\mathbb{R}} J_t(x, u) f''(du) < \infty \]

the series $f^J(x)$ is absolutely convergent. In both cases the series $f^J(x)$ is defined\(^4\) and satisfies

\[(2.13) \quad J^f_t(x) = \int_{\mathbb{R}} J_t(x, u) f''(du), \quad t \in [0, T].\]

**Proof.** From (2.11) we get

\[(2.14) \quad J^f_t(x_s, x_{s-}) = \int_{\mathbb{R}} |x_s - u| 1_{[x_{s-}, x_s]}(u) f''(du).\]

If $f$ is convex the series (2.2) defining $J^f_t(x)$ consists only of positive terms, and the thesis follows from (2.14), summing over $s \leq t$ and applying Fubini’s theorem. If instead $f = g - h$ with $g, h$ convex, observe that $f'' = g'' + h''$ and $\int_{\mathbb{R}} J_t(x, u) f''(du) < \infty$ (since $J_t(x, \cdot) = 0$ outside a compact), so (2.13) follows again from Fubini’s theorem; the absolute convergence of the series (2.2) follows writing

\[ |\Delta f(x_s) - f'(x_{s-})\Delta x_s| \leq (\Delta g(x_s) - g'(x_{s-})\Delta x_s) + (\Delta h(x_s) - h'(x_{s-})\Delta x_s),\]

summing the latter over $s \leq t$ and applying (2.13) to $g$ and $h$. \(\square\)

**Remark 2.10.** It follows from Lemma 2.9 and Hölder’s inequality that, if $J_t(x, \cdot) \in L^p(\mathbb{R})$ and $f''(du) = f''(u) du$ with $f'' \in L^q(\mathbb{R})$, where $p, q \geq 1$ are conjugate exponents, that is satisfy $1/p + 1/q = 1$, then the series (2.2) defining $J^f_t(x)$ is absolutely convergent. Moreover, if $J_t(x, \cdot)$ is bounded\(^3\) then the series (2.2) is absolutely convergent for every $f$ which is a difference of convex functions: indeed, $J_t(x, \cdot) = 0$ outside a compact, and $|f''| < \infty$ for every compact $C \subseteq \mathbb{R}$.

Here a more intuitive but cumbersome way of getting (2.14): define

\[ g(\cdot) := |x_s - |1_{[x_{s-}, x_s]}(\cdot)|,\]

\(^4\)See 2.1.

\(^3\)This happens for example if $\sum_{s \leq t} |\Delta x_s| < \infty$, by Remark 2.8.
which is in $L^1(\mathbb{R})$, equals zero outside a compact, and has derivatives

$$Dg = (\Delta x_s)\delta_{x_s} - 1_{[x_s,\infty)} + 1_{[x_s,\infty)}, \quad D^2g = (\Delta x_s)D\delta_{x_s} - \delta_{x_s} + \delta_{x_s}.$$ 

Then equation (2.14) is simply\(^6\) the identity $\int_{\mathbb{R}} f(u) (D^2g)(du) = \int_{\mathbb{R}} g(u) (D^2 f)(du)$.

**Proof of Proposition 2.5.** The series (2.2) defining $J^n_t(x)$ is absolutely convergent by Remark 2.10.\(^7\) If $h \in C^2(\mathbb{R})$, from Föllmer’s pathwise Itô formula (2.3), the definition of occupation time $L$ and of $K := L/2 + J$, and Lemma 2.9 it follows that

$$(2.15) \quad \int_{(0,t]} h'(x_s) \, dx_s = h(x_t) - h(x_0) - \int_{\mathbb{R}} K_t(x,u) \, h''(du), \quad t \in [0,T],$$

holds with $\int_{0}^{t} h'(x_u) \, dx_u$ defined via (2.4). Applying (2.15) to $h = f_n \in C^2(\mathbb{R})$ and taking lim$_n$ gives the thesis, since the right-hand side converges to

$$f(x_t) - f(x_0) - \int_{\mathbb{R}} K_t(x,u) f''(du)$$

because $K_t(x,\cdot) \in L^p(\mathbb{R})$ and $f_n \to f$ in $W^{2,q}(\mathbb{R})$ (so $f_n \to f$ pointwise and $f_n'' \to f''$ in $L^q(\mathbb{R})$), and so also the LHS converges. □

**Remark 2.11.** As in the stochastic setting, it follows from Tanaka–Meyer’s formula that $dL_t(u)$ is carried by the set $\{s \in (0,t] : x_s = x_{s-} = u\}$. Obviously, $dJ_t(u)$ is carried by the set

$$\{s \in (0,t] : u \in (x_{s-}, x_s) \text{ or } u \in (x_s, x_{s-})\}$$

of times at which $x$ jumps across level $u$.

**Remark 2.12.** It follows from (2.13) that, whenever Tanaka’s formula holds, it can be written as

$$f(x_t) - f(x_0) = \int_{(0,t]} f'(x_u) \, dx_u + \int_{\mathbb{R}} K_t(x,u) f''(du), \quad t \in [0,T],$$

where we recall that $K_t(u) := L_t(u)/2 + J_t(u)$. While uncommon, writing (2.16) seems more elegant than writing (2.9).

**Remark 2.13.** One can recover a continuous version $\mathcal{L}$ of the occupation time $L$ from knowing just a jointly measurable function $K_t(u)$ such that $K(u)$ is càdlàg increasing for a.e. $u$, $K_0 = 0$, $K_T \in L^1(\mathbb{R})$, and (2.16) holds for all $f \in C^2$ with $\int_{0}^{t} f'(x_u) \, dx_u$ defined via (2.4); it follows that the occupation time $L$ admits a continuous version $\mathcal{L}$ (thus its càdlàg version is actually continuous). Indeed, $\mathcal{L}(u)$ (resp. $J(u)$) is the continuous (resp. purely discontinuous) part of the increasing càdlàg function $K(u)$. To show this, consider that for $f \in C^2(\mathbb{R})$ Föllmer’s formula (2.3), (2.15) and Lemma 2.9 give that

$$K_t^f := \int_{\mathbb{R}} K_t^f(u) f''(du) + \int_{\mathbb{R}} K_t^d(u) f''(du) \, du = \frac{1}{2} \int_{0}^{t} f''(x_s) \, dx_s^c + \int_{\mathbb{R}} J_t(u) f''(du),$$

\(^6\)This equality holds a priori only when $f$ is $C^\infty(\mathbb{R})$ (by definition of distributional derivatives). However, with some work it follows that it holds for any $f$ which equals the difference of convex functions: indeed, since $g$ is càdlàg, convolving against a mollifier with support in $[0, \infty)$ shows that there exist $f_\epsilon \in C^\infty(\mathbb{R})$ such that $f_\epsilon \to f$ uniformly on compacts and $f \, g(u)(D^2f_\epsilon)(u) \, du \to f \, g(u)(D^2f)$, as shown in [DOS18, Proof of Theorem 5.2].

\(^7\)More precisely, if the jump is downward, then $x$ is allowed to jump from $x_{s-} = u$. 


where $K^c$ (resp. $K^d$) denotes the continuous (resp. purely discontinuous) part of $K(u)$. In each of the two above representations of the càdlàg increasing function $K^I_t$, the first term is continuous and the second purely discontinuous, so by uniqueness of such decomposition
\[
\int_{\mathbb{R}} K^I_t(u) g(u) \, du = \frac{1}{2} \int_{0}^{t} g(x_s) \, d[x]^c_s, \quad \int_{\mathbb{R}} K^d_t(u) g(u) \, du = \int_{\mathbb{R}} J_t(u) g(u) \, du
\]
holds for any $g$ of the form $f''$, i.e. for any continuous $g$; but then it also automatically holds for any Borel $g$, so $2K^c$ is an occupation local time of $x$ and $J_t = K^d_t$ a.e. $u$ for each $t$; since $J_t$ and $K^d_t$ are càdlàg in $t$, $J_t = K^d_t$ a.e. $u$ for all $t$.

Remark 2.14. For continuous paths $x$ the above approximation argument, without relying on the representation (2.12) can be used to obtain space-time Tanaka–Meyer formulae, see [FZ06]. Although elaborated in a probabilistic framework, the proofs in [FZ06] are (primarily) of pathwise nature.

Remark 2.15. The definition of occupation local times and the generalization of Itô’s formula to only twice weakly differentiable functions in Proposition 2.5 is based on Föllmer’s notion of quadratic variation and his pathwise Itô formula (Theorem 2.2). However, the Föllmer–Itô formula is by no means the only pathwise Itô-type formula, which can be extended to an Tanaka–Meyer formula in the spirit of in Proposition 2.5. For example, one could also start from the pathwise Itô formula based on càdlàg rough paths ([FZ18, Theorem 2.12]) or the one based on truncated variation ([Loc19, Theorem 4.1]) and proceed in an analogous manner as done in the present subsection.

Remark 2.16. If $x$ has an occupation local time $L$, then one can give explicit formulae for $L$. Indeed, since $L_t(\cdot) \in L^1(\mathbb{R})$, taking $\lim_{\epsilon \downarrow 0}$ of (2.5) applied to $g := 1_{[u-\epsilon,u+\epsilon]}$ gives that
\[
L_t(x, u) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_{0}^{t} 1_{[u-\epsilon,u+\epsilon]}(x_s) \, d[x]^c_s,
\]
meaning that the limit on the right-hand side exists and is a version of $L_t(\cdot)$. Analogously, if we can apply Tanaka-Meyer’s formula to the convex function $|\cdot - u|$ we get the following expression for $L$:
\[
(2.17) \quad L_t(x, u) = |x_t - u| - |x_0 - u| - \int_{[0,t]} \text{sign}(x_{s-} - u) \, dx_s - 2J_t(x, u), \quad t \in [0,T].
\]
It is then desirable to establish if (a version of) Proposition 2.5 holds in the case where $f: \mathbb{R} \to \mathbb{R}$ equals the difference of two convex functions. This is indeed the case, under the additional assumptions that the mollifier $\rho$ has compact support in $[0, \infty)$, that $J_t(u)$ is càdlàg in $u$ for all $t$ (see Remark 2.8), and that there exists a version $L_t$ of the pathwise local time $L_t$ which is càdlàg in $u$ for all $t$ (in particular, unlike in the stochastic setting, one cannot use (2.17) to prove that $L$ has a version which is càdlàg in $u$ for all $t$ without running into circular arguments). Indeed, under these assumptions the proof of [DOS18, Theorem 5.2] shows that $\int_{\mathbb{R}} g(u) f''(du) \rightarrow \int_{\mathbb{R}} g(u) f''(du)$ for any càdlàg $g$, and if we apply this to $g = K_t$ the rest of the proof of Proposition 2.5 goes through.

2.3. Local time via discretization. An alternative approach to achieve a pathwise Tanaka–Meyer formula goes back to Würlim [Wue80] and is based on a discrete version of the Tanaka–Meyer formula. For continuous paths $x$ this approach is well-understood and lead to several extensions, see [PPP15, DOS18, CP19]. One feature of this discretization argument is that
the “stochastic” integral $\int_0^t f'(x_s) \, dx_s$ is still given as a limit of left-point Riemann sums, see also [DOR14]. In the present subsection we generalize Würmil's approach to the case of càdlàg paths $x$.

A natural extension of Würmil's definition of local times of continuous paths to càdlàg paths reads as follows. Given a partition $\pi = (t_j)_{j=0}^n$ of $[0, T]$, define the discrete level crossing time of $x$ at $u$ (along $\pi$) as the function

$$K^\pi_t(x, u) := \sum_{t_i \in \pi} 1_{[x_{t_i}, x_{t_i+1} \wedge t]}(u)|x_{t_i+1} \wedge t - u|, \quad t \in [0, T].$$

Then, applying (2.11) to $a = t_i \wedge t, b = t_{i+1} \wedge t$ and summing over $i$, we obtain the discrete version of Tanaka–Meyer formula

$$f(x_t) - f(x_0) - \sum_{t_i \in \pi} f'(x_{t_i})(x_{t_{i+1} \wedge t} - x_{t_i \wedge t}) = \int_{\mathbb{R}} K^\pi_t(u) f''(du).$$

**Definition 2.17.** Let $x \in D([0, T]; \mathbb{R})$ and let $(\pi_n)_n$ partitions such that $|\pi_n| \to 0$. A function $K: [0, T] \times \mathbb{R} \to \mathbb{R}$ is called the $L^p$-level crossing time of $x$ (along $(\pi_n)_n$) if $K^\pi_t$ converges weakly in $L^p(\mathbb{R})$ to $K_t$ for each $t \in [0, T]$ as $n \to \infty$, and $t \mapsto \int_{\mathbb{R}} K_t(u) \, du$ is right-continuous.

The set $L^W_p((\pi_n)_n)$ denotes all paths $x \in D([0, T]; \mathbb{R})$ having a $L^p$-level crossing time along $(\pi_n)_n$.

In the case of continuous paths, the constructions of $K$ coincides with Würmil’s definition of local time for continuous paths as presented in, e.g., [DOR14] Definition B.3. To be precise, note there is one slight difference between these definitions: While Definition 2.17 and [DOR14] Definition B.3 require weak convergence, Würmil’ originals definition in [Wue80] actually asks for strong convergence of the discrete level crossing times.

**Lemma 2.18.** The level crossing time $K$ in Definition 2.17 is increasing in $t \in [0, T]$, i.e., $K_s(\cdot) \leq K_t(\cdot)$ a.e for each $s \leq t$.

**Proof.** Given $\pi = (t_j)_j$, let $m(\pi, s)$ be the value of $j$ such that $t_j < s \leq t_{j+1}$, and write

$$K^\pi_s = \sum_{j < m(\pi, s)} a_j(u) + 1_{[x_{m(\pi, s)}, x_s]}(u)|x_s - u|,$$

where $a_j(u) := 1_{[x_j, x_{j+1}]}(u)|x_{j+1} - u|$. If $s < t$, analogously write

$$K^\pi_t - \sum_{j < m(\pi, s)} a_j(u) - a_m(\pi, s)(u) = \sum_{m(\pi, s) < j < m(\pi, t)} a_j(u) + 1_{[x_{m(\pi, s)}, x_t]}(u)|x_t - u| =: R^\pi_{s, t}.$$ 

Thus

$$K^\pi_t - K^\pi_s - R^\pi_{s, t} = a_m(\pi, s)(u) - 1_{[x_{m(\pi, s)}, x_s]}(u)|x_s - u| =: S_s(\pi, u),$$

and since $R^\pi_{s, t} \geq 0$ the thesis follows once we prove that $S_s(\pi^n, u) \to 0$ for every $u$ when $|\pi^n| \to 0$. This holds since if $m(n) := m(\pi^n, s)$ then $t_{m(n)}$ and $t_{m(n)+1}$ converge to $s$, and $t_{m(n)} < s \leq t_{m(n)+1}$, so

$$a_m(\pi^n)(u) \quad \text{and} \quad 1_{[x_{m(n)}, x_s]}(u)|x_s - u|$$

both converge to $1_{[x_s, x_s]}(u)|x_s - u|$ as $n \to \infty$, since $x$ is càdlàg. \qed
Notice that $K_t$ is only defined as an equivalence class. As discussed before Remark 2.7, we can always choose for each $t$ a version of $K_t$ which is càdlàg increasing in $t$ for each $u$; from now on, we will always work with such version. If $K^c$ (resp. $K^d$) denotes the continuous (resp. purely discontinuous) part of the increasing càdlàg function $K(u)$, the following holds.

**Proposition 2.19.** Suppose that $x \in \mathcal{L}^W_p((\pi^n)_n)$ for $p, q \in [1, \infty]$ with $1/p + 1/q = 1$. If $f \in \mathcal{W}^{2,q}(\mathbb{R})$, then the following limit exists (and is finite)

\begin{equation}
\int_0^t f'(x_{s-}) \, dx_s := \lim_{n \to \infty} \sum_{t_i \in \pi^n} f'(x_i)(x_{i+1} - x_i), \quad t \in [0, T],
\end{equation}

and the pathwise Tanaka–Meyer formula (2.16) holds with such definition of $\int_0^t f'(x_{s-}) \, dx_s$. Moreover, $2K^c$ is the occupation local time of $x$ and $J_t(u)$ (given by (2.7)) equals $K^d(u)$ for a.e. $u$ for all $t$. In particular, also the pathwise Tanaka–Meyer formula (2.9) holds (with $L = 2K^c$), the two definitions (2.8) and (2.20) of $\int_0^t f'(x_{s-}) \, dx_s$ coincide, and $\mathcal{L}^W_p((\pi^n)_n) \subseteq \mathcal{L}^p((\pi^n)_n)$.

**Proof of Proposition 2.19.** Taking the limit as $n$ goes to $\infty$ of the discrete Tanaka–Meyer formula (2.19) applied to $\pi_n$, we obtain that the pathwise Tanaka–Meyer formula (2.16) holds if using the definition (2.20). Now, from Remark 2.13 it follows that $2K^c$ satisfies the occupation time formula and the rest follows as well. $\square$

**Remark 2.20.** Following the seminal paper [Fal81], we consider the “stochastic” integral as limit of left-point Riemann sums (2.20) and not as limit of

\begin{equation}
\sum_{t_i \in \pi^n(t)} f'(x_i)(x_{i+1} - x_i), \quad t \in [0, T].
\end{equation}

In a probabilistic setting, where $x$ is assumed to be a semimartingale, these limits coincide with the classical Itô integral almost surely (see [Pro04, Chapter II.5, Theorem 21]) and so they are equal. In the present pathwise setting however, they could be different.

**Remark 2.21.** Applying Minkowski’s integral inequality and using the identity (2.10), we obtain that if $p \in [1, \infty)$ and $C_p := 1/(p + 1)^{1/p}$, then

\begin{equation}
\|K^c_t\|_{L^p} \leq C_p \sum_{t_i \in \pi} |x_{i+1} - x_i|^{1 + \frac{1}{p}}.
\end{equation}

In particular, if $x \in \mathcal{L}^W_1((\pi^n)_n)$, then the occupation local time $L$ equals $2K^c$ and so satisfies

\begin{equation}
\|L_t\|_{L^p} \leq 2\|K_t\|_{L^p} \leq 2C_p \liminf_n \sum_{t_i \in \pi_n} |x_{i+1} - x_i|^{1 + \frac{1}{p}} \quad \text{for every } p \in [1, \infty).
\end{equation}

**Remark 2.22.** Given the definition of $J_t(u)$, it seems natural that, if $x \in \mathcal{L}^W_p((\pi^n)_n)$ and

\begin{equation}
J^p_t(u) := \sum_{t_i \in \pi^n(t)} 1_{\{x_{i-} < x_i\}}(u)|x_{i-} - u|, \quad u \in \mathbb{R}, \ t \in [0, T],
\end{equation}

then $J^p_t$ also converges weakly in $\mathcal{L}^p(\mathbb{R})$. If we assume this and denote by $L_t^d$ the limit, if $(\pi^n)_n$ are refining and $\cup_n \pi^n \supseteq \{s \in [0, T] : \Delta x_s \neq 0\}$ then $L_t^d = J = K^d$ a.e. (and so $K_t^c - J_t^n$ converges weakly in $\mathcal{L}^p(\mathbb{R})$ to $K_t^d$). Indeed, (2.11) gives

\begin{equation}
J^p_t := \sum_{t_i \in \pi^n(t)} f(x_i) - f(x_{i-}) - f'(x_{i-}) (x_i - x_{i-}) = \int_{(\mathbb{R})} J^p_t(u) f''(u) \, du,
\end{equation}
and thanks to Lemma [2.9] we can apply the dominated convergence theorem to conclude that 
\( J_t^{n,π} \to J_t^f \), so taking \( n \to \infty \) in (2.22) we get 
\[
J_t^f = \int_\mathbb{R} L_t^f(u)f''(u)\,du,
\]
for every \( f \in L^4 \), so by Lemma [2.9] \( L_t^f = J_t \) a.e.

2.4. Local time via normalized number of interval crossings. Instead of regularization
the function \( f \) appearing in the “stochastic” integral \( \int_0^t f'(x_{s-})\,dx_s \), one could alternatively
approximate the path \( x \) by sufficiently regular functions \( (x^n) \), ensuring that the “stochastic”
integral \( \int_0^t f'(x^n_{s-})\,dx_s \) is well-defined for each \( x^n \).

We implement this idea based on so-called double Skorokhod problem. This choice
of approximation has the additional feature that it leads to a nice interpretation of the obtained
local time in terms of interval crossings. Let \( V^1([0, T]; \mathbb{R}) \subset D([0, T]; \mathbb{R}) \) and \( V^+([0, T]; \mathbb{R}) \subset D([0, T]; \mathbb{R}) \) be the space of all functions with bounded variation (also called of finite total
variation) and of all non-decreasing functions, respectively. Let us recall that for \([0, t] \subset [0, T] \)
and \( y: [0, T] \to \mathbb{R} \), the total variation of \( y \) on the interval \([0, t] \) is given by

\[
TV(y, [0, t]) := \sup \left\{ \sum_{i=0}^{N-1} |y_{t_{i+1}} - y_{t_i}| : (t_i)_{i=0}^N \text{ is a partition of } [0, t], N \in \mathbb{N} \right\}.
\]

**Definition 2.23.** Given \( x \in D([0, T]; \mathbb{R}) \) and \( c > 0 \), a pair
\( (\phi^c, -x^c) \in D([0, T]; \mathbb{R}) \times V^1([0, T]; \mathbb{R}) \) is called a solution to the Skorokhod problem on \([-c/2, c/2] \) if the following
conditions are satisfied:

(i) \( x_t - x_t^c = \phi_t^c \in [-c/2, c/2] \) for every \( t \in [0, T] \),

(ii) \( x^c = x_u^c - x_u^d \) with \( x_u^c, x_u^d \in V^+([0, T]; \mathbb{R}) \subset D([0, T]; \mathbb{R}) \) and the corresponding measures
\( dx_u^c \) and \( dx_u^d \) are supported in \( \{ t \in [0, T] : \phi_t^c = c/2 \} \) and \( \{ t \in [0, T] : \phi_t^c = -c/2 \} \), respectively,

(iii) \( \phi_0^c = 0 \).

The Skorokhod problem is well-studied in the literature, see, e.g., [KLRS07, BKR09]. In
particular, the above Skorokhod problem (Definition 2.23) has a unique solution, see [LG14
Proposition 2.7]. Let us emphasize that for any \( c > 0 \), \( x^c \) is a càdlàg path of bounded variation,
which uniformly approximates \( x \) with accuracy \( c/2 \). Hence, keeping in mind integration by
parts for the Lebesgue–Stieltjes integral, the integral
\[
\int_0^t f'(x_{s-})\,dx_t, \quad t \in [0, T],
\]
(recall that we apply the convention \( f'_0 = f'_0(\cdot, 0) \)) is well-defined for every \( c > 0 \) and \( f \in W^{2,q}(\mathbb{R}) \), where \( q \geq 1 \). More precisely, we define it in the following way

\[
\int_0^t f'(x_{s-})\,dx_s := f'(x_t^c)x_t - f'(x_0^c)x_0 - \int_0^t x_s\,df'(x_s^c) - \sum_{0 < s \leq \xi} \Delta x_s \Delta f'(x_s^c),
\]
where \( \int_0^t x_s\,df'(x_s^c) \) exists as the Lebesgue–Stieltjes integral.

Given \( x \in D([0, T]; \mathbb{R}) \) and a sequence \( (c_n) \) such that \( c_n > 0 \) and \( c_n \to 0 \) we define the
pair \( (\phi^n, -x^n) \) (abusing slightly previously used notation) as the solution to the Skorokhod
problem on \([-c_n/2, c_n/2] \) for \( n \in \mathbb{N} \) and approximate \( x \) in the following by the sequence \( (x^n) \).
Starting from this approximation, the corresponding notion of pathwise local time can be defined as normalized limits of the numbers of interval crossings. For \( z \in \mathbb{R}, c > 0 \) and \( t \in (0, T] \) we define the number of upcrossings of a path \( x \in D([0, T]; \mathbb{R}) \) of the interval \([z - c/2, z + c/2]\) over the time \([0, t]\) by

\[
\nu^{z,c}(x, [0, t]) := \sup_{n \in \mathbb{N}} \sup_{0 \leq t_1 < s_1 < \cdots < s_n \leq t} \sum_{i=1}^{n} 1_{\{x_{i-1} \leq z-c/2 \text{ and } x_i > z+c/2\}}
\]

The number of downcrossings \( \nu^{z,c}(x, [0, t]) \) is defined analogously. We set

\[
\nu^{z,c}(x, [0, t]) := \nu^{z,c}(x, [0, t]) + \nu^{z,c}(x, [0, t])
\]

for the total number of crossings.

**Definition 2.24.** Let \( x \in D([0, T]; \mathbb{R}) \) and \((c_n)_n\) be a sequence such that \( c_n > 0 \) and \( c_n \to 0 \). A function \( L: [0, T] \times \mathbb{R} \to \mathbb{R} \) is called \( L^p \)-local time of \( x \) along \((c_n)_n\) if, for all \( t \in [0, T] \),

\[
(1) \quad c_n \cdot \nu^{z,c_n}(x, [0, t]), \quad z \in \mathbb{R},
\]

converges weakly in \( L^p(\mathbb{R}; \mathbb{R}) \) to \( L_t \) as \( n \to \infty \),

(2) and

\[
(2) \quad J_t(x^{c_n}, z), \quad z \in \mathbb{R},
\]

converges weakly in \( L^p(\mathbb{R}; \mathbb{R}) \) to \( J_t(x, \cdot) \) as \( n \to \infty \).

The set \( \mathcal{L}^2_p((c_n)_n) \) denotes all paths \( x \in D([0, T]; \mathbb{R}) \) having an \( L^p \)-local time along \((c_n)_n\) in the above sense.

The corresponding pathwise Tanaka–Meyer formula reads as follows.

**Proposition 2.25.** Suppose that \( x \in \mathcal{L}^2_p((c_n)_n) \) for \( p, q \geq 1 \) with \( 1/p + 1/q = 1 \). If \( f \in W^{2,q}(\mathbb{R}) \), then the following limit exists (and is finite)

\[
\int_0^t f'(x_{s-}) \, dx_s := \lim_{n \to \infty} \int_0^t f'(x_{s-}^{c_n}) \, dx_s, \quad t \in [0, T],
\]

and the pathwise Tanaka–Meyer formula (2.9) holds with such definition of \( \int_0^t f'(x_{s-}) \, dx_s \) and with \( J_t^f(x) \) as given in (2.2).

**Proof.** W.l.o.g. we may assume that \( f \in W^{2,q}(\mathbb{R}) \) is of the form

\[
(2.24) \quad f(x) = (|\cdot| * f^{''})(x), \quad x \in \mathbb{R},
\]

where \( f'' \) has compact support and \( f' \) is continuous and non-decreasing. Indeed, since \( x \) is càdlàg, we can assume w.l.o.g. that \( f \) has compact support. The representation (2.24) follows from (2.12), keeping in mind that Proposition 2.25 is trivial for affine functions. Finally, \( f' \) is continuous and of bounded variation, as \( f \in W^{2,q}(\mathbb{R}) \), which also ensures that we can decompose it in the difference of two non-decreasing functions.

Let us consider the integral \( \int_0^t f'(x_{s-}^{c_n}) \, dx_s \). For \( t \in [0, T] \) we have

\[
(2.25) \quad \int_0^t f'(x_{s-}^{c_n}) \, dx_s = f'(x_{t}^{c_n}) x_t - f'(x_0^{c_n}) x_0 - \int_0^t x_{s-} \, df'(x_{s}^{c_n}) - \sum_{0 < s \leq t} \Delta x_s \Delta f'(x_{s}^{c_n}),
\]
where \( \int_0^t x_s \, d f'(x^n_s) \) is the Lebesgue–Stieltjes integral. Further, we have

\[
\int_0^t x_s \, d f'(x^n_s) + \sum_{0 < s \leq t} \Delta x_s \Delta f'(x^n_s) = \int_0^t x_s \, d f'(x^n_s)
\]

(2.26)

\[
= \int_0^t \{x_s - x^n_s\} \, d f'(x^n_s) + \int_0^t x^n_s \, d f'(x^n_s).
\]

To calculate the first integral one may use the properties of the double Skorokhod problem (and the fact that \( f' \) is non-decreasing and \( x^n \) is piecewise monotonic) and notice that for \( s \in (0, t] \) we have

\[
\text{if } d f'(x^n_s) > 0 \text{ then } x_s - x^n_s = c_n/2;
\]

\[
\text{if } d f'(x^n_s) < 0 \text{ then } x_s - x^n_s = -c_n/2.
\]

By \( d f'(x^n_s) > 0 \) for \( s \in (0, t] \) we mean that \( \Delta f'(x^n_s) > 0 \) or that \( \Delta f'(x^n_s) = 0 \) but for any \( \varepsilon > 0 \) which is sufficiently close to 0, \( f'(x^n_s) \) is non-decreasing and non-constant on \( (s - \varepsilon, s] \) and non-decreasing on \([s, s + \varepsilon) \cap (0, t] \), or \( f'(x^n_s) \) is non-decreasing on \( (s - \varepsilon, s] \) and non-constant and non-decreasing on \([s, s + \varepsilon) \cap (0, t] \). Thus

\[
\int_0^t \{x_s - x^n_s\} \, d f'(x^n_s) = \frac{c_n}{2} \text{TV}(f'(x^n), [0, t]) .
\]

Using (2.25), (2.26) and (2.27), we finally arrive at

\[
\int_0^t f'(x^n_s) \, dx_s = f'(x^n_0)x_t - f'(x^n_0)x_0 - \int_0^t x^n_s \, d f'(x^n_s) + \frac{c_n}{2} \text{TV}(f'(x^n), [0, t]),
\]

(2.28)

where \( \text{TV}(f'(x^n), [0, t]) \) is the total variation of \( f'(x^n) \) on the interval \([0, t] \) as defined in (2.23).

The right side of (2.27) may be also calculated using the Banach indicatrix theorem, using the numbers of level crossings:

\[
\text{TV}(f'(x^n), [0, t]) = \int \mathcal{N}^y(f'(x^n), [0, t]) \, dy = \int \mathcal{N}^z(x^n, [0, t]) \, df'(z),
\]

(2.29)

where \( \mathcal{N}^y(y, [0, t]) \) is the number of crossings the level \( y \) by càdlàg \( g \) (as defined in Remark 1.3).

Now we will deal with \( \int_0^t x^s_n \, df'(x^n_s) \). Since \( x^n \) and \( f'(x^n) \) have finite total variation the rules of the Lebesgue–Stieltjes integral (integration by parts and the substitution rule) apply here and we have

\[
\int_0^t x^n_s \, df'(x^n_s) = f'(x^n_t)x^n_t - f'(x^n_0)x^n_0 - \int_0^t f'(x^n_s) \, dx^n_s + \sum_{0 < s \leq t} \Delta x^n_s \Delta f'(x^n_s)
\]

(2.30)

and

\[
\int_0^t f'(x^n_s) \, dx^n_s = f(x^n_t) - f(x^n_0) - \sum_{0 < s \leq t} (\Delta f(x^n_s) - f'(x^n_s)\Delta x^n_s).
\]

(2.31)

Now, from (2.28), (2.30), (2.31) and (2.29) we get

\[
f(x^n_t) - f(x^n_0) = \int_0^t f'(x^n_s) \, dx_s + \frac{c_n}{2} \int \mathcal{N}^z(x^n, [0, t]) \, df'(z) + J_f(x^n)
\]

\[\quad - f'(x^n_t)(x_t - x^n_t) + f'(x^n_0)(x_0 - x^n_0).\]

(2.32)
For all \( z \in \mathbb{R} \) except a countable set, the numbers \( u^{z,c}(x,[0,t]) \), \( d^{z,c}(x,[0,t]) \) and \( n^{z,c}(x,[0,t]) \) are equal the numbers of (up-, down-) crosses of the value interval \((z - c/2; z + c/2)\) (Remark 1.4). Since \( N^z(x^n,[0,t]) = n^{z,c}(x,[0,t]) = 0 \) when \( z < \inf_{s \in [0,t]} x_s - c_n/2 \) or \( z > \sup_{s \in [0,t]} x_s + c_n/2 \) and \( N^z(x^n,[0,t]) = n^{z,c}(x,[0,t]) \pm 1 \) for other \( z \) (see [LG14] Lemma 3.3 and 3.4), we have that

\[
\lim_{n \to +\infty} c_n \int_{\mathbb{R}} n^{z,c}(x,[0,t]) \, df'(z) = \lim_{n \to +\infty} \int_{\mathbb{R}} c_n \cdot n^{z,c}(x,[0,t]) \, f''(z) \, dz = \int_{\mathbb{R}} L_t(z) f''(z) \, dz,
\]

where the last equality follows from the first assumption in Definition 2.24. Also, by the second assumption in Definition 2.24.

\[
\lim_{n \to +\infty} J^f_t(x^n) = \lim_{n \to +\infty} \int_{\mathbb{R}} J_t(x^n, y) f''(y) \, dy = \int_{\mathbb{R}} J_t(x, y) f''(y) \, dy = J^f_t(x).
\]

The last two limits together with (2.32) give the thesis. \( \square \)

**Remark 2.26.** To apply Proposition 2.25 we need to know when \( c_n \cdot n^{z,c}(x,[0,t]) \) and \( J_t(x^n,\cdot) \) converge weakly in \( L^p(\mathbb{R}) \) to some \( L_t, J_t(x,\cdot) \in L^p(\mathbb{R}) \). It is even not clear when \( c_n \cdot n^{z,c}(x,[0,t]) \) and \( J_t(x^n,\cdot) \) belong to \( L^p(\mathbb{R}) \). Here we give some general criteria. If for some \( r > 0, V^r([0,0],[0,t]) < \infty \), where \( V^r \) denotes \( r \)-variation defined for \( y;[0,T] \to \mathbb{R} \) and \( t \in [0,T] \) as

\[
V^r(y,[0,t]) := \sup \left\{ \sum_{i=0}^{N-1} |y_{t_{i+1}} - y_{t_i}|^r : (t_i)_{i=0}^N \text{ is a partition of } [0,t], N \in \mathbb{N} \right\},
\]

then \( c_n \cdot n^{z,c}(x,[0,t]) \) is bounded (and is equal 0 outside a compact subset of \( \mathbb{R} \)), thus belongs to \( L^p(\mathbb{R}) \). It follows from the easy estimate: for any \( z \in \mathbb{R} \)

\[
n^{z,c}(x,[0,t]) \leq \frac{V^r(x,[0,t])}{c_n^r}.
\]

Unfortunately, this observation does not yield any condition which guarantees \( L_t \in L^p(\mathbb{R}) \) except rather trivial case \( r \leq 1 \).

Similarly as in Remark 2.6 we have that if \( p \in [1,\infty) \) and \( \sum_{0<s \leq t} |\Delta x_s|^{1+1/p} < \infty \) then \( J_t(x^n,\cdot), J_t(x,\cdot) \in L^p(\mathbb{R}) \). This follows from Minkowski’s inequality and the fact that for any \( s > 0, |\Delta x_s^n| \leq |\Delta x_s| \) (see [LG14] (2.5) or [Loc14] Section 2). Next, using Hölder’s inequality, for \( q \) such that \( 1/p + 1/q = 1 \) and any \( f \in W^{2,q}(\mathbb{R}), s \in [0,t] \) we obtain

\[
\int_{\mathbb{R}} |x^n_s - u|^{1+1/q} \, df(u) \, du \leq C_p |\Delta x^n_s|^{1+1/p} \, ||f''||_{L^q} \leq C_p |\Delta x_s|^{1+1/p} \, ||f''||_{L^q}.
\]

Using this estimate and the dominated convergence theorem we obtain weak convergence of \( J_t(x^n,\cdot) \) to \( J_t(x,\cdot) \) in \( L^p(\mathbb{R}) \).

**3. Construction of local times for càdlàg semimartingales**

The purpose of this section is to give probabilistic constructions of the pathwise local time, as provided in Definition 2.4, 2.17 and 2.23 for càdlàg semimartingales. In particular, almost surely all definitions coincide with the classical definition of local times in the case of càdlàg semimartingales.
3.1. Local times via discretisation and as occupation measure. Given a càdlàg semimartingale \( X = (X_t)_{t \in [0, \infty)} \) on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and \( u \in \mathbb{R} \), one can define \( J_t(u)(\omega) := J_t(X(\omega), u) \), with \( J_t(x, u) \) given by (2.7), and the increasing càdlàg adapted process \( K(u) \) by
\[
2K_t(u) := |X_t - u| - |X_0 - u| - \int_{[0,t]} \text{sign}(X_{s^-} - u) \, dX_s.
\]

It can then be shown that there exists a jointly measurable version of \( K_t(u, \omega) \) such that the family of processes \( \mathcal{L} = 2K - 2J \), called the (classical) local time of \( X \), satisfies the Tanaka–Meyer formula (2.9), is càdlàg in \( t \) and is jointly measurable: see\footnote{Recall the identity (2.6) and notice that the notations used in [Pro04] differ from ours: he calls \( A^u \) what we call \( 2\mathcal{K}(u) \).} [Pro04, Chapter 4, Section 7]. In the following we denote by \( L^p(\mu) \) the \( L^p \)-space with respect to the measure \( \mu \).

The following is the main theorem of this section.

**Theorem 3.1.** Assume that \( f: \mathbb{R} \to \mathbb{R} \) is the difference of two convex functions, that \( \pi_n \) are optional partitions of \([0, \infty)\) such that \( |\pi_n \cap [0, t]| \to 0 \) a.s. for all \( t \) and that \( X = (X_t)_{t \in [0, \infty)} \) is a càdlàg semimartingale. Then, there exists a subsequence \( (n_k)_k \) such that, for \( \omega \) outside a \( \mathbb{P} \)-null set (which may depend on \( f^n \)),
\[
\sup_{s \in [0,t]} \left| K_s^{\pi_{n_k}}(X,\omega)(u) - K_s(\omega, u) \right| \to 0 \quad \text{in } L^p([f^n](du)) \quad \text{as } k \to \infty
\]

simultaneously for all \( p \in [1, \infty) \), \( t < \infty \), where we recall the definition \( K_s^{\pi_n}(X,\omega)(u) \) was given by (2.18).

**Remark 3.2.** Theorem 3.1 essentially says that the pathwise crossing time \( K_s^{\pi_n}(X, u) \) sampled along optional partitions \( (\pi_n)_n \) (defined applying (2.18) to each path \( X(\omega) \) and partition \( \pi_n(\omega) \)) converges to \( K(u) \). Applying Theorem 3.1 with \( f(x) = x^2/2 \) gives in particular that a.e. path of a càdlàg semimartingale is in \( L^p([\pi_{n_k}]_k) \subset L^p(\pi_{n_k})_k \) for all \( p < \infty \), i.e. the \( L^p \)-level crossing time and the occupation local time exist for a.e. path of a semimartingale. Indeed, \( K_t^{\pi_{n_k}}(X, \cdot) \to K_t(\cdot) \) strongly (and thus weakly) in \( L^p(\mathbb{R}) \) for a.e. \( \omega \), locally uniformly in \( t \).

To prove the previous theorem we need some preliminaries. Given \( p \in [1, \infty) \) we denote by \( \mathcal{S}^p \) the set of càdlàg special semimartingales \( X \) which satisfy
\[
\|X\|_{\mathcal{S}^p} := \left\| [M]^{1/2} \right\|_{L^p(\mathbb{P})} + \left\| \int_0^\infty d|V|_t \right\|_{L^p(\mathbb{P})} < \infty,
\]
where \( X = M + V \) is the canonical semimartingale decomposition of \( X \), \( [M]_t := M_t^2 - 2 \int_0^t M_s \, dM_s \) is the quadratic variation of the martingale \( M \), and \( |V|_t \) is the variation up to time \( t \) of the predictable process \((V_t)_{t \in [0,\infty)} \). We recall the existence of \( c_p < \infty \) such that the inequality
\[
\|\sup_t |X_t|\|_{L^p(\mathbb{P})} \leq c_p\|X\|_{\mathcal{S}^p},
\]
holds for all local martingales \( X \) (this being one side of the Burkholder–Davis–Gundy inequalities), and thus also trivially extends to all \( X \in \mathcal{S}^p \). The core of Theorem 3.1 is the following more technical statement.
Proposition 3.3. Let $\pi_n$ be optional partitions of $[0, \infty)$ such that $|\pi_n \cap [0,t]| \to 0$ a.s. for all $t$. If $X \in \mathcal{S}^p$ for $p \in [1, \infty)$, and

$$h_{\pi_n}(u) := \left\| \sup_{t \in [0,\infty)} K_{t}^{\pi_n}(\omega, u) - K_{t}^{\pi_n}(\omega, u) \right\|_{L^p(\mathbb{P})}, \ u \in \mathbb{R}.$$ 

then, for every $u \in \mathbb{R}$, $h_{\pi_n}(u) \to 0$ as $n \to \infty$ and $0 \leq h_{\pi_n}(u) \leq c_p\|X\|_{\mathcal{S}^p}$ for all $n \in \mathbb{N}$.

As discussed in detail in [DOS18] after Theorem 6.2, for a continuous process $X$ and properly chosen $\pi_n$ the convergence of $K_{\pi_n}^{\pi_n}(X, u)$ is closely related to the number of upcrossings of $X$ from the level $u$ to the level $u + \varepsilon > u$. While stronger versions of the above theorems have already appeared in the case of continuous semimartingales (the strongest being [Lem83, Theorem II.2.4]), in the càdlàg setting we were only able to locate in the literature a version of Theorem 3.1 where, under the strong assumption that $\sum_{n \leq t} |\Delta X_n| < \infty$ a.s., the $L^p(|f^m|(du))$ convergence is replaced by pointwise convergence for all but countably many values of $u$, see [Lem83, Theorem III.3.3]. Thus, compared to the literature, our method provides an unusually strong conclusion, with the benefit of a simple proof. Other differences are that we consider the crossing time instead of the number of upcrossings, and we use partitions such that $|\pi_n| \to 0$ instead of “Lebesgue partitions” (in the language of [DOS18]).

Proof of Proposition 3.3. Consider the convex function $f(x) := |x - u|$ and let us take its left-derivative $\text{sign}(x-u)$ and its second (distributional) derivative $2\delta_u$. Subtracting from the discrete-time Tanaka–Meyer formula (2.19) its continuous-time stochastic counterpart (3.1) and considering the process $K_{\pi_n}^{\pi_n}(u)(\omega) := K_{\pi_n}^{\pi_n}(X, u)$, we obtain

$$0 = \int_0^t (H_s^{\pi_n}(u) - H_s(u)) \, dX_s + 2(K_s^{\pi_n}(u) - K_s(u)),$$

where for $\pi_n = (\tau_{\pi_i}^n, i)$ by $H_{\pi_n}$ and $H(u)$ we denote the predictable processes

$$H_{\pi_i}^{\pi_n}(u) := \sum_i \text{sign}(X_{\tau_i}^n - u)\mathbf{1}_{(\tau_i^n, \tau_{i+1}^n]}(s) \quad \text{and} \quad H_s(u) := \text{sign}(X_{s-} - u).$$

Now $h_{\pi_n}(u) \to 0$ for each $u \in \mathbb{R}$ follows from (3.2) and (3.3) if we show that $\int_0^t H_{\pi_i}^{\pi_n}(u) \, dX_s \to \int_0^t H_s(u) \, dX_s$ in $\mathcal{S}^p$. To this end fix $n$ and $u$ and notice that from

$$H_{\pi_i}^{\pi_n}(u) = \text{sign}(X_{\tau_i}^n - u), \quad \text{for } i \text{ such that } \tau_i^n < s \leq \tau_{i+1}^n$$

and $|\pi_n \cap [0,t]| \to 0$ a.s. for all $t$ it follows that $H_{\pi_i}^{\pi_n}(u) \to H_s(u)$ a.s. for all $s$. Since $|H_{\pi_i}^{\pi_n}(u) - H_s(u)| \leq 2$ it follows that $\int_0^t H_{\pi_i}^{\pi_n}(u) \, dX_s \to \int_0^t H_s(u) \, dX_s$ in $\mathcal{S}^p$ (by the dominated convergence theorem) and that

$$h_{\pi_n}(u) \leq \frac{c_p}{2} \left\| \int_0^t (H_s^{\pi_n}(u) - H_s(u)) \, dX_s \right\|_{\mathcal{S}^p} \leq c_p\|X\|_{\mathcal{S}^p} \quad \text{for all } u \in \mathbb{R},$$

concluding the proof.

Proof of Theorem 3.1. Let $(\tau_m)_m$ a sequence of stopping times which prelocalizes $X$ to $\mathcal{S}^p$ (see Emery [Eme79, Theoreme 2]), i.e. $\tau_m \uparrow \infty$ a.s. and $X^{\tau_m -} \in \mathcal{S}^p$ for all $m$. Let $\mu_i(A) := |f^m|(A \cap [-i, i])$ and set

$$G_n(\omega, T, u) := \sup_{t \in [0, T]} |K_{t}^{\pi_n}(\omega, u) - K_{t}^{\pi_n}(\omega, u)|$$

and $G_{\pi_n} := \mathbf{1}_{(T < \tau_m)} G_n$. Since $\mu_i$ is a finite measure, Proposition 3.3 implies that, as $n \to \infty$, $G_{\pi_n}$ converges to 0 in $L^p(\mathbb{P} \times \mu_i)$, for all $m, i \in \mathbb{N}$ and $T \geq 0$. Passing to a subsequence
Local times via segment crossings. Let us recall the definition of $L^p$-local time of a deterministic path along a sequence of positive reals tending to 0 (Definition 2.24). In this subsection we will prove the following theorem.

**Theorem 3.4.** Let $X = (X_t)_{t \in [0, \infty)}$ be a càdlàg semimartingale and $T > 0$. There exist a $\mathbb{P}$-null set $E$ such that for any $\omega \in \Omega \setminus E$ and any sequence of positive reals $(c_n)_n$ which converge to 0, $x_t = X_t(\omega)$, $t \in [0, T]$, belongs to $\mathcal{L}_t^1((c_n)_n)$ and for any $t \in [0, T]$ the $L^1$-local time of $x$ along $(c_n)_n$, $L_t$, coincides (in $L^1(\mathbb{R})$) with the classical local time of $X$, $\mathcal{L}_t(u)$, was proven in Lemma 3.3, see Lemma 3.3, Theorem III.3.3, but only for semimartingales whose jumps are a.s. summable, that is $\sum_{0 < s \leq t} |\Delta X_s| < \infty$ for any $t > 0$.

Before we prove this result we will need some more definitions and auxiliary results of pathwise nature, which may be of independent interest. Let $V^0([0, \infty); \mathbb{R})$ denote the subset of $D([0, \infty); \mathbb{R})$ of piecewise monotonic càdlàg paths $x: [0, \infty) \to \mathbb{R}$, that is such functions $x \in D([0, \infty); \mathbb{R})$ that for any $T > 0$ there exist finite number of intervals $I_i$, $i = 1, 2, \ldots, N$, $N \in \mathbb{N}$, such that $\bigcup_{i=1}^N I_i = [0, T]$ and $x$ is monotonic on each $I_i$. Let us notice that $V^0([0, \infty); \mathbb{R}) \subset V^1([0, \infty); \mathbb{R})$, where $V^1([0, \infty); \mathbb{R})$ was defined in Subsection 2.4. Similarly to the total variation, for $y: [0, T] \to \mathbb{R}$ we define its upward total variation on the interval $[0, t] \subset [0, T]$ as

$$\text{UTV}(y, [0, t]) := \sup \left\{ \sum_{i=0}^{N-1} \max(y_{t_{i+1}} - y_{t_i}, 0) : (t_i)_{i=0}^N \text{ is a partition of } [0, t], N \in \mathbb{N} \right\}$$

as well as its downward total variation on the interval $[0, t]$ as

$$\text{DTV}(y, [0, t]) := \sup \left\{ \sum_{i=0}^{N-1} \max(y_{t_i} - y_{t_{i+1}}, 0) : (t_i)_{i=0}^N \text{ is a partition of } [0, t], N \in \mathbb{N} \right\}.$$

Let us recall numbers of interval (up-, down-) crossings $u^{\Delta^+}(z, [0, t])$, $d^{\Delta^-}(z, [0, t])$ and $n^{\Delta^+}(z, [0, t])$ we define level (up, down-) crossings.

**Definition 3.5.** Let $z \in \mathbb{R}$. The number of times that the function $x: [0, \infty) \to \mathbb{R}$ upcrosses the level $z$ over the time $[0, t]$ is defined as

$$u^+(x, [0, t]) = \lim_{c \to 0} u^{\Delta^+}(z, [0, t]).$$

Analogously we define the number of downcrosses as

$$d^-(x, [0, t]) = \lim_{c \to 0} d^{\Delta^-}(x, [0, t]).$$
Lemma 3.6. Let \( x \in V^0[0, \infty), \ t > 0 \) and \( g : \mathbb{R} \to \mathbb{R} \) be locally bounded and Borel-measurable. Then

\[
(3.4) \quad \int_{\mathbb{R}} g(z) u^\gamma(x, [0, t]) \, dz = \int_0^t g(x_{s-}) \text{UTV}(x, ds) + \sum_{0 < s \leq t, \Delta x > 0} \int_{x_{s-}}^{x_s} [g(z) - g(x_{s-})] \, dz,
\]

and analogously

\[
(3.5) \quad \int_{\mathbb{R}} g(z) d^\gamma(x, [0, t]) \, dz = \int_0^t g(x_{s-}) \text{DTV}(x, ds) + \sum_{0 < s \leq t, \Delta x < 0} \int_{x_{s-}}^{x_s} [g(z) - g(x_{s-})] \, dz.
\]

Proof. By the assumption \( x \in V^0([0, \infty); \mathbb{R}) \) there exists a finite sequence of intervals \( \{I_i\}_{i=1}^N, N \in \mathbb{N}, \) which are mutually disjoint, \( \bigcup_{i=1}^N I_i = [0, t] \) and the function \( x \) is monotone on any of \( I_i. \)

Since \( x \) is càdlàg, we may and will assume that these intervals, except the last one containing \( t, \) are of the form \( I_i = [t_i, t_{i+1}), \) where \( t_i < t_{i+1}. \) Moreover, we will assume that they are the largest intervals possible on which \( x \) is monotone. Thus, if for some \( i > 1, x \) is non-decreasing on \( [t_{i-1}, t_i) \) and non-decreasing on \( [t_i, t_{i+1}) \) then there need to be a downward jump at the time \( t_i. \) Similarly, if for some \( i > 1, x \) is non-increasing on \( [t_{i-1}, t_i) \) and non-increasing on \( [t_i, t_{i+1}) \) then there need to be an upward jump at the time \( t_i. \) Let \( S \) denote the set of such times \( t_i. \)

Let us define \( R = \sum_{i=1}^N \{x_{t_i}, x_{t_{i+1}}\}. \) Given a level \( z \in \mathbb{R} \setminus R, \) with any interval \( I_i \) or any \( s \in S \) we can have at most one associated upcrossing. This happens when \( x_{t_i} < x_{t_{i+1}} \) or \( x_{s-} < x_s \) and

\[
z \in (x_{t_i}, x_{t_{i+1}}) \quad \text{or} \quad z \in (x_{s-}, x_s).
\]

Clearly, neither of intervals where function is decreasing, induces any upcrossing. Let \( J \) be the subset of the indices where the function is increasing and \( T \) be the subset of \( s \in S \) such that \( x \) has upward jump at \( s. \) Then we have

\[
u^\gamma(x, [0, t]) = \sum_{i \in J} 1(x_{t_i}, x_{t_{i+1}})(z) + \sum_{s \in T} 1(x_{s-}, x_s)(z).
\]

Consequently

\[
\int_{\mathbb{R}} g(z) u^\gamma(x, t) \, dz = \int_{\mathbb{R} \setminus R} g(z) u^\gamma(x, t) \, dz = \sum_{i \in J} \int_{x_{t_i}}^{x_{t_{i+1}}} g(z) \, dz + \sum_{s \in T} \int_{x_{s-}}^{x_s} g(z) \, dz.
\]

We are now to deal with these integrals. To this end we apply the classical idea of opening temporal windows at times of jumps. We may impose that the sum of the lengths of these windows is finite and consider interpolated continuous \( \tilde{x}. \) We have

\[
\int_{x_{t_i}}^{x_{t_{i+1}}} g(z) \, dz = \int_{\tilde{x}_{u}}^{\tilde{x}_{u}'} g(z) \, dz \quad \text{and} \quad \int_{x_{s-}}^{x_s} g(z) \, dz = \int_{\tilde{x}_v}^{\tilde{x}_v'} g(z) \, dz
\]

for properly defined \( u, u', v, v'. \) Then we apply the change of variable for the Riemann–Stieltjes integral

\[
\int_{\tilde{x}_u}^{\tilde{x}_u'} g(z) \, dz = \int_{\tilde{x}_u}^{\tilde{x}_u'} g(\tilde{x}_s) \, d\tilde{x}_s \quad \text{and} \quad \int_{\tilde{x}_v}^{\tilde{x}_v'} g(z) \, dz = \int_{\tilde{x}_v}^{\tilde{x}_v'} g(\tilde{x}_s) \, d\tilde{x}_s
\]

Clearly, for properly defined \( \tilde{t} \) we have

\[
(3.6) \quad \int_{\mathbb{R}} g(z) u^\gamma(x, t) \, dz = \int_0^T g(\tilde{x}_s) \text{UTV}(\tilde{x}, ds).
\]
Let us consider the decomposition of the measure $\text{UTV}(x, ds)$ into the continuous part $\mu_c$ and the atomic part $\mu_a$. Moreover, let $W$ be the union of the temporal windows. We obtain
\[
\int_0^t 1_{s \in W} g(\tilde{x}_s) \text{UTV}(\tilde{x}, ds) = \int_0^t g(x_{s-}) \, d\mu_c(s),
\]
möreover
\[
\int_0^t 1_{s \in W} g(\tilde{x}_s) \text{UTV}(\tilde{x}, ds) = \sum_{0 < s \leq t, \Delta x_s > 0} \int_{x_{s-}}^{x_s} g(z) \, dz.
\]
Using this and (3.6) we obtain (3.4) by simple calculations. In a similar way one proves (3.5).

To state the next result we will need to define functionals called upward truncated variation and downward truncated variation with the truncation parameter $c \geq 0$. For $y \in D([0, \infty); \mathbb{R})$ they are defined similarly as the upward and downward total variations, but changes of the values of $y$ which are smaller than $c$ are neglected. The upward truncated variation of $y$ on the interval $[0, t]$, with the truncation parameter $c$ is given by
\[
\text{UTV}^c(y, [0, t]) := \sup \left\{ \sum_{i=0}^{N-1} \max(y_{t_{i+1}} - y_{t_i} - c, 0) : (t_i)_{i=0}^N \text{ is a partition of } [0, t], N \in \mathbb{N} \right\}
\]
while the downward truncated variation of $y$ on the interval $[0, t]$, with the truncation parameter $c$ is given by
\[
\text{DTV}^c(y, [0, t]) := \sup \left\{ \sum_{i=0}^{N-1} \max(y_{t_i} - y_{t_{i+1}} - c, 0) : (t_i)_{i=0}^N \text{ is a partition of } [0, t], N \in \mathbb{N} \right\}.
\]

Using Lemma 3.6 we will prove the following theorem, relating the integrated number of interval crossings of a function $x \in D([0, \infty); \mathbb{R})$ with the asymptotics of the upward and downward truncated variations $\text{UTV}^c(x, t), \text{DTV}^c(x, t), t \geq 0$, as $c \to 0$.

**Theorem 3.7.** Let $x \in D([0, \infty); \mathbb{R})$ Assume that there exist a function $\varphi : [0, \infty) \to [0, \infty)$, such that $\lim_{c \to 0} \varphi(c) = 0$, and a continuous function $\zeta : [0, \infty) \to \mathbb{R}$, such that for $t \geq 0$:
\[
\varphi(c) \text{UTV}^c(x, [0, t]) \to \frac{1}{2} \zeta_t \quad \text{as } c \to 0
\]
and
\[
\varphi(c) \text{DTV}^c(x, [0, t]) \to \frac{1}{2} \zeta_t \quad \text{as } c \to 0.
\]
Then $\zeta$ is non-negative, non-decreasing and for any continuous function $g : \mathbb{R} \to \mathbb{R}$ we have the following convergences
\[
\varphi(c) \int_{\mathbb{R}} n^{x,c}(x, [0, \cdot]) g(z) \, dz \to \frac{1}{2} \int_0^t g(x_{t-}) \, d\zeta_t,
\]
and
\[
\varphi(c) \int_{\mathbb{R}} d^{x,c}(x, [0, \cdot]) g(z) \, dz \to \frac{1}{2} \int_0^t g(x_{t-}) \, d\zeta_t
\]
in the uniform convergence topology on compact subsets of positive half-line $[0, \infty)$. 

Remark 3.8. Theorem 3.7 remains valid when the convergence \( c \to 0 \) is replaced by a convergence along some sequence \( c_n \to 0 \) as \( n \to \infty \). To see this it is enough to replace everywhere in the proof \( c \) by \( c_n \).

Proof. Assume that (3.7) holds. The fact that \( \zeta \) is non-negative and non-decreasing follows immediately from the same properties of the function \( t \mapsto \text{UTV}^c(x, [0, t]) \), \( t \geq 0 \).

Now let us fix \( T > 0 \) and put

\[
M = \max \left\{ \sup_{0 \leq t \leq T} |x_t|, \sup_{0 \leq t \leq T} |\zeta_t| = \zeta_T \right\}.
\]

Using the assumption that \( g \) is continuous, for fixed \( \varepsilon > 0 \) we can find a number \( \delta > 0 \) such that

\[
(3.9) \quad \sup_{z, z' \in [-M-1; M+1], |z-z'| \leq \delta} |g(z) - g(z')| \leq \frac{\varepsilon}{M}.
\]

Let us also define

\[
N = \# \{ t \in (0; T) : |x_t - x_{t-}| > \delta \}.
\]

Let \( t \in [0; T] \). We will use the regularisation \( x^c \) of \( x \) obtained via Skorokhod map. Since for any \( z \in \mathbb{R} \) and any \( t > 0 \), \( u^c(x^c, [0, t]) = u^c(x^c, [0, t]) \pm 1 \) ([LG14, Lemma 3.3 and 3.4]) and \( u^c(x^c, [0, t]) = u^c(x^c, [0, t]) = 0 \) when \( z < \inf_{c \in [0, t]} x_s - c/2 \) or \( z > \sup_{s \in [0, t]} x_s + c/2 \) to prove convergence (3.8) it is sufficient to prove the convergence

\[
\varphi(c) \int_{\mathbb{R}} u^c(x^c, [0, \cdot]) g(z) \, dz \to \frac{1}{2} \int_0^t g(x_{t-}) \, d\zeta_t \quad \text{as} \quad c \to 0.
\]

To prove this convergence will use Lemma 3.6. From the properties of the Skorokhod map we have that \( x^c \) is piecewise monotonic (see [LG14, Proposition 2.7 and formula (2.4)]). Using Lemma 3.6 we write

\[
(3.10) \quad \int_{\mathbb{R}} g(z) u^c(x^c, [0, t]) \, dz = \int_0^t g(x_{t-}) \text{UTV}^+(x^c, ds) \sum_{0 < s \leq t, \Delta x_s \geq \delta} \int_{x_s^-}^{x_s^+} |g(z) - g(x_{s-}^c)| \, dz
\]

\[
+ \sum_{0 < s \leq t, 0 < \Delta x_s \leq \delta} \int_{x_s^-}^{x_s^+} |g(z) - g(x_{s-}^c)| \, dz.
\]

Let us denote the consecutive summands on the right side of equation (3.10) by \( L_1(c) \), \( L_2(c) \) and \( L_3(c) \), respectively.

First, for \( c > 0 \) we estimate

\[
\varphi(c) L_2(c) \leq \varphi(c) \cdot 2N(2M + c) \left( 2 \sup_{z \in [-M-c/2; M+c/2]} |g(z)| \right) \to 0 \quad \text{as} \quad c \to 0.
\]

Using (3.9) and the fact that \( \text{UTV}(x^c, [0, t]) \leq \text{UTV}^c(x, [0, t]) + c \) (see [LG14, Proposition 2.9]), we estimate

\[
\varphi(c) L_3(c) \leq \varphi(c) \sum_{0 < s \leq t, 0 < \Delta x_s \leq \delta} |\Delta x_s| \sup_{z, z' \in [-M-c/2; M+c/2], |z-z'| \leq \delta} |g(z) - g(z')|
\]

\[
\leq \varphi(c) \text{UTV}(x^c, [0, t]) \frac{\varepsilon}{M} \leq \varphi(c)(\text{UTV}^c(x, [0, t]) + c) \frac{\varepsilon}{M} \leq 2\varepsilon
\]

for \( c \) small enough such that \( \varphi(c) \text{UTV}^c(x, [0, T]) + \varphi(c)c \leq 2M \).
Now we are left with $L_1$. We fix $K = 1, 2, \ldots$ and define

$$I_i = \left[ -M + 2M \frac{i - 1}{K}; -M + 2M \frac{i}{K} \right], \quad i = 1, 2, \ldots, K.$$ 

Further we define the following sequence $\{v_k\}_{k \geq 0}$ of times. Let $v_0 = 0$ and

$$v_k = \inf \{ t > v_{k-1} : x_t \in I_i \text{ for some } i = 1, 2, \ldots, K \text{ such that } x_{v_{k-1}} \notin I_i \}.$$ 

We have

$$L_1(c) = \sum_{k=0}^{\infty} \int_{(v_k \wedge t, v_{k+1} \wedge t]} \left( g(x^c_{s-}) - g(x^c_s) \right) \text{UTV}(x^c, ds)$$

$$= \sum_{k=0}^{\infty} \int_{(v_k \wedge t, v_{k+1} \wedge t]} \left\{ g(x^c_{s-}) - g(x^c_s) \right\} \text{UTV}(x^c, ds)$$

$$+ \sum_{k=0}^{+\infty} g(x^c_{v_k \wedge t}) \text{UTV}(x^c, (v_k \wedge t, v_{k+1} \wedge t]),$$

(3.11)

where we denote $\text{UTV}(x^c, (v_k \wedge t, v_{k+1} \wedge t)) = \text{UTV}(x^c, [0, v_{k+1} \wedge t]) - \text{UTV}(x^c, [0, v_k \wedge t])$. Using (3.9) it is easy to estimate for $c \leq 2\delta$ and $2M/K \leq \delta$ the first summand in (3.11) multiplied by $\varphi(c)$:

$$\varphi(c) \sum_{k=0}^{\infty} \int_{(v_k \wedge t, v_{k+1} \wedge t]} \left| g(x^c_{s-}) - g(x^c_s) + g(x^c_s) - g(x^c_{v_k \wedge t}) \right| \text{UTV}(x^c, ds)$$

$$\leq \varphi(c) \sum_{k=0}^{\infty} \frac{2\varepsilon}{M} \text{UTV}(x^c, (v_k \wedge t, v_{k+1} \wedge t]) \leq \varphi(c) \text{UTV}(x^c, [0, t]) \frac{2\varepsilon}{M}$$

$$\leq \varphi(c) (\text{UTV}^c(x, [0, t]) + c) \frac{2\varepsilon}{M} \leq 4\varepsilon$$

for $c$ small enough such that $\varphi(c) \text{UTV}^c(x, [0, T]) + \varphi(c)c \leq 2M$. Now we investigate the second summand of $L_1$ multiplied by $\varphi(c)$:

$$\sum_{k=0}^{\infty} g(x^c_{v_k \wedge t}) \varphi(c) \text{UTV}(x^c, (v_k \wedge t, v_{k+1} \wedge t)) \to \frac{1}{2} \sum_{k=0}^{\infty} g(x^c_{v_k \wedge t}) \{\zeta_{v_{k+1} \wedge t} - \zeta_{v_k \wedge t}\}$$

as $c \to 0$. Since for each $t \in [0, t]$ there is only finite number of $k = 0, 1, \ldots$ for which $v_k \leq t$, the convergence in (3.12) is uniform on $[0, T]$. Moreover for $2M/K \leq \delta$

$$\left| \int_{[0,t]} g(x_{s-}) \, d\zeta_s - \sum_{k=0}^{\infty} g(x^c_{v_k \wedge t}) \{\zeta_{v_{k+1} \wedge t} - \zeta_{v_k \wedge t}\} \right|$$

$$= \sum_{k=0}^{+\infty} \int_{(v_k \wedge t, v_{k+1} \wedge t]} \left\{ g(x^c_{s-}) - g(x^c_{v_k \wedge t}) \right\} \, d\zeta_s$$

$$\leq \sum_{k=0}^{+\infty} \int_{(v_k \wedge t, v_{k+1} \wedge t]} \frac{\varepsilon}{M} \, d\zeta_s = \frac{\varepsilon}{M} \zeta_T \leq \frac{\varepsilon}{M} \zeta_T \leq \varepsilon.$$

(3.13)
Proof of Theorem 3.4. We get
\[ \sup_{0 \leq t \leq T} \left| \varphi(c) \int_{\mathbb{R}} g(z)u^{z,c}(f, t) \, dz - \frac{1}{2} \int_{[0,t]} g(x_{s-}) \, d\zeta_s \right| \leq 10\varepsilon \]
for any \( c > 0 \) sufficiently close to 0.

The proof of the convergence of \( \int_{\mathbb{R}} g(z)dz^{z,c}(x, [0, t]) \, dz \) is analogous. The convergence of \( \int_{\mathbb{R}} g(z)n^{z,c}(x, [0, t]) \, dz \) follows from the relation \( n^{z,c}(x, [0, t]) = u^{z,c}(x, [0, t]) + dz^{z,c}(x, [0, t]) \) and the convergences of \( \int_{\mathbb{R}} g(z)u^{z,c}(x, [0, t]) \, dz \) and \( \int_{\mathbb{R}} g(z)dz^{z,c}(x, [0, t]) \, dz \).

**Remark 3.9.** Under the assumptions of Theorem 3.7, from the facts that \( \zeta \) is continuous and that each càdlàg path has countable number of jumps, we also get uniform (on compacts) convergences
\[ \varphi(c) \int_{\mathbb{R}} u^{z,c}(x, [0, \cdot]) g(z) \, dz \to \frac{1}{2} \int_0^t g(x_t) \, d\zeta_t, \]
\[ \varphi(c) \int_{\mathbb{R}} d^{z,c}(x, [0, \cdot]) g(z) \, dz \to \frac{1}{2} \int_0^t g(x_t) \, d\zeta_t, \]
\[ \varphi(c) \int_{\mathbb{R}} n^{z,c}(x, [0, \cdot]) g(z) \, dz \to \int_0^t g(x_t) \, d\zeta_t. \]

Now we are ready to prove Theorem 3.4.

**Proof of Theorem 3.4.** By [Loc19] Theorem 2.1, there exists a \( \mathbb{P} \)-null set \( E_1 \) such that for any \( \omega \in \Omega \setminus E_1 \), \( x = X(\omega) \) satisfies assumptions of Theorem 3.7 with \( \varphi(c) \equiv c \) and \( \zeta_t = [X]_{\text{cont}}^t(\omega) \),

where \( [X]_{\text{cont}}^t \) denotes the continuous part of the quadratic variation of \( X \).

From this observation, Remark 3.8 and 3.9, the fact that classical local time \( L_t \) satisfies occupation times formula (2.5), and from monotone class argument with help of Remark 2.26 we get that for all \( \omega \in \Omega \setminus E_1 \) such that \( [X]_T(\omega) < \infty \) and \( V^3(X(\omega), [0, T]) < \infty \) (where \( V^r \) denotes \( r \)-variation defined in Remark 2.26) the weak limit of \( c_n \cdot u^{z,c}(x, [0, t]) \) in \( L^1(\mathbb{R}) \) exists and is equal \( L_t \). Let use denote \( \Omega_2 = \Omega \setminus \{ \omega \in \Omega : [X]_T(\omega) < \infty \} \cap \{ \omega \in \Omega : V^3(X(\omega), [0, T]) < \infty \} \). We have \( \mathbb{P}(\Omega_2) = 1 \) (the fact that \( V^3(X(\omega), [0, T]) < \infty \) with probability 1 follows for example from [Lep76 Théorème 1]).

For \( \omega \in \Omega_2 \) we also have \( \sum_{0 \leq s \leq t} (\Delta X_s(\omega))^2 \leq \infty \). This observation together with Remark 2.26 yields that if \( \omega \in \Omega_2 \) and \( x = X(\omega) \) then the sequence \( (J_t(x^n, \cdot))_n \) converges weakly in \( L^1(\mathbb{R}) \) to \( J_t(x, \cdot) \).

Thus we proved that for \( \omega \in \Omega_2 \) and \( x = X(\omega) \) both required (weak) convergences hold, thus \( x \in \mathcal{L}^1((c_n)_n) \).

**References**


