Stock returns modelling— from classical model to models with jumps and extreme events

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Uncertain behaviour of stock returns

- The price of a stock (for example the price of one share of Cemex noted on *Bolsa Mexicana de Valores*) is not known in advance
  - $S_t$ - price today (or at time $t$)
  - $S_{t+1}$ - price tomorrow (or at time $t+1$ in the future)
- **Return** – how many pesos more will produce tomorrow (or in the future – at time $t+1$) one peso invested in a Cemex share today (at time $t$)?
Return cont.

• If we had invested K pesos then we could buy \( \frac{K}{S_t} \) shares.

• If we sell them at time \( t+1 \), for \( S_{t+1} \) pesos each, then we will get \( S_{t+1} \frac{K}{S_t} \) pesos.

• Thus one peso produces
  \[
  \frac{S_{t+1}K}{S_t} - 1 = \frac{S_{t+1}}{S_t} - 1
  \]
  pesos more (if it is negative, then it means that we have lost some pesos ;-( )
Return cont.

- The return between times $t$ and $t+1$ reads as

$$R_{t,t+1} = \frac{S_{t+1}}{S_t} - 1$$

- since $S_{t+1}$ is unknown the return is unknown too

- Even if it is unknown, one may have some expectation about the future price – based on up-to-date observations
Randomness – the simplest model

• The most simple model assumes that
  – with probability 0.5 the return will be equal a
  – with probability 0.5 the return will be equal -b

Stock price = $8

0.5

Stock Price = $8(1+a)

0.5

Stock Price = $8(1-b)
The simplest model cont.

- What will be the stock price after 3 days?
- Possible scenarios

Stock price = $8

\[
\begin{align*}
$8(1+a) & \quad 0.5 \\
$8(1-b) & \quad 0.5
\end{align*}
\]

\[
\begin{align*}
$8(1+a)^2 & \quad 0.5 \\
$8(1+a)(1-b) & \quad 0.5
\end{align*}
\]

\[
\begin{align*}
$8(1-b)^2 & \quad 0.5
\end{align*}
\]

\[
\begin{align*}
$8(1+b)^3 & \quad 0.5 \\
$8(1+b)^2(1-b) & \quad 0.5 \\
$8(1-b)^2(1+a) & \quad 0.5 \\
$8(1-b)^3 & \quad 0.5
\end{align*}
\]
The simplest model cont.

- What will be the stock price after $n$ days?
Real prices
The simplest model vs. real prices

• At the first sight the simplest model seems to provide good approximation of real prices movements
  – the only problem is the choice of the parameters a and b

• But in reality the returns between consecutive days are not always equal +a or –b for some a and b

• They attain a whole spectrum of values
Example – recent returns of Nikkei Index

<table>
<thead>
<tr>
<th>Date</th>
<th>Last</th>
<th>Open</th>
<th>High</th>
<th>Low</th>
<th>Return %</th>
</tr>
</thead>
<tbody>
<tr>
<td>03/23/2011</td>
<td>9449,47</td>
<td>9553</td>
<td>9568</td>
<td>9376,5</td>
<td>-1,65%</td>
</tr>
<tr>
<td>03/22/2011</td>
<td>9608,32</td>
<td>9545</td>
<td>9610,5</td>
<td>9455,5</td>
<td>4,36%</td>
</tr>
<tr>
<td>03/18/2011</td>
<td>9206,75</td>
<td>9083</td>
<td>9260,5</td>
<td>9075</td>
<td>2,72%</td>
</tr>
<tr>
<td>03/17/2011</td>
<td>8962,67</td>
<td>8543</td>
<td>9080,5</td>
<td>8443</td>
<td>-1,44%</td>
</tr>
<tr>
<td>03/16/2011</td>
<td>9093,72</td>
<td>9105</td>
<td>9135,5</td>
<td>8782,5</td>
<td>5,07%</td>
</tr>
<tr>
<td>03/15/2011</td>
<td>8655</td>
<td>9235</td>
<td>9287,5</td>
<td>7862,5</td>
<td>-10,04%</td>
</tr>
<tr>
<td>03/14/2011</td>
<td>9620,49</td>
<td>9575</td>
<td>9817,5</td>
<td>9520</td>
<td>-6,18%</td>
</tr>
<tr>
<td>03/11/2011</td>
<td>10254,43</td>
<td>10350</td>
<td>10375,5</td>
<td>10225,5</td>
<td>-1,72%</td>
</tr>
<tr>
<td>03/10/2011</td>
<td>10434,38</td>
<td>10528</td>
<td>10543,5</td>
<td>10400,5</td>
<td>-1,46%</td>
</tr>
<tr>
<td>03/09/2011</td>
<td>10589,5</td>
<td>10615</td>
<td>10660,5</td>
<td>10565,5</td>
<td>0,61%</td>
</tr>
<tr>
<td>03/08/2011</td>
<td>10525,19</td>
<td>10520</td>
<td>10565,5</td>
<td>10505,5</td>
<td>0,19%</td>
</tr>
<tr>
<td>03/07/2011</td>
<td>10505,02</td>
<td>10610</td>
<td>10615</td>
<td>10465,5</td>
<td>-1,76%</td>
</tr>
</tbody>
</table>
Closer look at the frequency of daily returns of Nikkei Index since March 2010.
The need of the refinement of the model

• The need of the refinement follows from significant frequency of the whole spectrum of returns

• The idea – what if we change the prices more frequently, but follow the same pattern?

• In fact the changes of stock and index prices occur every hour or even every minute!
Classical model - refinement of the simplest model

- A bit more sophisticated mathematical considerations lead to the following classical model

- Natural logarithm

\[ \ln(1 + R_{t,t+1}) \]

(sometimes called logarithmic return) is a random quantity (mathematicians say: random variable) with normal distribution
Normal distribution

• Normal distribution has two parameters – **mean** and **variance** – and knowing them **we are able to calculate probability** that
  - the daily return will be higher than a certain threshold
  or
  - the daily return will be lower than a certain threshold
Normal distribution - example

Density function of normal distribution
Normal distribution - probabilities

• If $\ln(1 + R_{t,t+1})$ has mean $\mu$ and variance $\sigma^2$ than the probability that $R_{t,t+1}$ will be smaller than some $R$ may be calculated with a bit complicated formula

$$P \left( R_{t,t+1} \leq R \right) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\ln(1+R)} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx$$

• but it is simply the area of the light-blue figure on the previous slide
Calculation of probabilities in the classical model

- For daily returns (since March 2010) of the Nikkei Index we have estimated
  \( \mu \approx -0.0005, \sigma \approx 0.016 \)
- and calculated the probabilities

\[
\begin{align*}
P(R_{t,t+1} \leq \mu - 4\sigma) &= P(R_{t,t+1} \geq \mu + 4\sigma) \approx 0.00003 \\
P(R_{t,t+1} \leq \mu - 3\sigma) &= P(R_{t,t+1} \geq \mu + 3\sigma) \approx 0.00135 \\
P(R_{t,t+1} \leq \mu - 2\sigma) &= P(R_{t,t+1} \geq \mu + 2\sigma) \approx 0.02275 \\
P(R_{t,t+1} \leq \mu - \sigma) &= P(R_{t,t+1} \geq \mu + \sigma) \approx 0.15865
\end{align*}
\]
Probabilities as the measures of risk

• The knowledge of the calculated probabilities is important in risk management
• They tell a stockholder for what loss he/she shall be prepared to
• Since they are important they have even their own name – Value at Risk
• In classical model they depend only on the parameters $\mu$ and $\sigma$
Probabilities - estimation

- Since there were approximately 250 trading days on Tokio Stock Exchange, the probability that the daily return will be higher than a certain threshold $R$ may be estimated as:

\[
\frac{\text{number of days with bigger return than } R}{250}
\]

- Similarly, the probability that the daily return will be lower than a certain threshold $R$ reads:

\[
\frac{\text{number of days with smaller return than } R}{250}
\]
## Probabilities - comparison

<table>
<thead>
<tr>
<th>Threshold R</th>
<th>Theoretical probability return &lt;R</th>
<th>Number of days return &lt;R</th>
<th>Estimated probability return &lt;R</th>
</tr>
</thead>
<tbody>
<tr>
<td>-6.23%</td>
<td>0.00003</td>
<td>1</td>
<td>0.00408</td>
</tr>
<tr>
<td>-4.68%</td>
<td>0.00135</td>
<td>2</td>
<td>0.00816</td>
</tr>
<tr>
<td>-3.14%</td>
<td>0.02275</td>
<td>5</td>
<td>0.02041</td>
</tr>
<tr>
<td>-1.59%</td>
<td>0.15866</td>
<td>32</td>
<td>0.13061</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Threshold R</th>
<th>Theoretical probability return &gt;R</th>
<th>Number of days return &gt;R</th>
<th>Estimated probability return &gt;R</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.23%</td>
<td>0.00003</td>
<td>0</td>
<td>0.00000</td>
</tr>
<tr>
<td>4.68%</td>
<td>0.00135</td>
<td>1</td>
<td>0.00408</td>
</tr>
<tr>
<td>3.14%</td>
<td>0.02275</td>
<td>3</td>
<td>0.01224</td>
</tr>
<tr>
<td>1.59%</td>
<td>0.15866</td>
<td>28</td>
<td>0.11429</td>
</tr>
</tbody>
</table>
Problem with the classical model – extreme events

• The classical model poorly predicts occurrence of extreme events (such as the occurrence of the daily return of Nikkei Index lower than -6% - due to the earthquake and tsunami in Japan)

• Such events occur more often than classical model predicts, and what is even more problematic – they may cause much bigger loses than „usual” events
Extreme events – help!!!

• Great help in modelling extreme events provides to us modern Extreme Value Theory (EVT)

• It gives general framework to estimate probabilities of extreme values

• The idea is to choose a certain thresholds $R_- < 0, R_+ > 0$ and to model separately returns
  – between $R_-$ and $R_+$ – with classical model
  – smaller than $R_-$ or greater than $R_+$ – with EVT
Probabilities of extreme values

• The formula for probability of extreme return, greater than $R_+ + u$, given by EVT reads as the following product

$$ P(R_{t,t+1} \geq R_+ + u) = \left(1 + \frac{\xi u}{\beta_+}\right)^{-1/\xi} \times P(R_{t,t+1} \geq R_+) $$

• where probability $P\left(R_{t,t+1} \geq R_+\right)$ is calculated with classical model

• Similarly we deal with left tail probabilities, i.e.

$$ P(R_{t,t+1} \leq R_- - u) $$
EVT – what we need to use it?

• To use EVT we need to fix here three more parameters: $R_+ > 0, \xi_+ > 0, \beta_+ > 0$ - threshold, shape and scale parameters

• Similarly, we need three more parameters to calculate left tail probabilities

• It may be a bit difficult, because we have small number of observations exceeding thresholds

• But once it is done, we have much realistic prediction of probabilities of extreme values
EVT for Nikkei Index

- For Nikkei Index we obtain the following estimates based on 10 years observations

\[ P \left( R_{t,t+1} \geq 4\% + u \mid R_{t,t+1} \geq 4\% \right) \approx \left( 1 + 25 \cdot u \right)^{-4} \]

- Below we have comparison of the probabilities calculated with this formula and real frequencies

<table>
<thead>
<tr>
<th>( u )</th>
<th>real frequency ( R&gt;4%+u ) among all 34 returns &gt;4%</th>
<th>calculated frequency ( R&gt;4%+u ) among all 34 returns &gt;4%</th>
</tr>
</thead>
<tbody>
<tr>
<td>0,50%</td>
<td>0.6</td>
<td>0,624295077</td>
</tr>
<tr>
<td>1%</td>
<td>0.5</td>
<td>0,4096</td>
</tr>
<tr>
<td>1,50%</td>
<td>0.32</td>
<td>0,279762311</td>
</tr>
<tr>
<td>2%</td>
<td>0.2</td>
<td>0,197530864</td>
</tr>
<tr>
<td>3%</td>
<td>0.15</td>
<td>0,106622241</td>
</tr>
</tbody>
</table>
EVT for Nikkei Index, cont.

- For risk measurement it is more important to calculate probabilities of the adverse market movements, and they are associated with negative returns. We have

\[ P\left( R_{t,t+1} \leq -2\% - u \mid R_{t,t+1} \leq -2\% \right) \approx \left( 1 + 28.75 \cdot u \right)^{-4.35} \]

<table>
<thead>
<tr>
<th>( u )</th>
<th>real frequency ( R&lt;-2%\text{-}u ) among all 310 returns ( &lt;-2% )</th>
<th>calculated frequency ( R&lt;-2%\text{-}u ) among all returns ( &lt;-2% )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50%</td>
<td>0.545</td>
<td>0.557519015</td>
</tr>
<tr>
<td>1%</td>
<td>0.335</td>
<td>0.333119086</td>
</tr>
<tr>
<td>1.50%</td>
<td>0.2</td>
<td>0.210202355</td>
</tr>
<tr>
<td>2%</td>
<td>0.145</td>
<td>0.138621174</td>
</tr>
<tr>
<td>3%</td>
<td>0.087</td>
<td>0.066846885</td>
</tr>
<tr>
<td>4%</td>
<td>0.038</td>
<td>0.035800797</td>
</tr>
<tr>
<td>5%</td>
<td>0.0226</td>
<td>0.020739147</td>
</tr>
<tr>
<td>6%</td>
<td>0.0129</td>
<td>0.012768965</td>
</tr>
</tbody>
</table>
Other models – models with jumps

• There are other approaches possible to deal with extreme events – one of them is to allow jumps in simulations of stock prices
• This approach uses more advanced mathematical models
• They may seem difficult at the first sight but the mathematics in their background is very beautiful
Model with jumps
Classical model
Real quotes
Thank you!