On pathwise stochastic integration with finite variation processes uniformly approximating càdlàg processes

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Truncated variation - how it appears

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Question: what is the smallest total variation possible of a (càdlàg) function from the ball $\{ g : \| f - g \|_\infty \leq \frac{1}{2} c \}$?
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\inf_{g : \| f - g \|_\infty \leq \frac{1}{2} c} TV (g, [a; b]) \geq TV^c (f, [a; b]),
\]

where

\[
TV (g, [a; b]) := \sup_n \sup_{a \leq t_0 < t_1 < \ldots < t_n \leq b} \sum_{i=1}^{n} |g(t_i) - g(t_{i-1})|,
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TV^c (f, [a; b]) := \sup_n \sup_{a \leq t_0 < t_1 < \ldots < t_n \leq b} \sum_{i=1}^{n} \max \{|f(t_i) - f(t_{i-1})| - c, 0\},
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and follows immediately from the inequality

\[
|g (t_i) - g (t_{i-1})| \geq \max \{ |f (t_i) - f (t_{i-1})| - c, 0 \}.
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truncated variation of the function $f$ at the level $c$. 
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Truncated variation may be interpreted not only as the lower bound obtained on the previous slide but also as the variation taking into account only jumps greater than $c$. 
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**truncated variation** of the function $f$ at the level $c$.

Truncated variation may be interpreted not only as the lower bound obtained on the previous slide but also as the variation taking into account only jumps greater than $c$.

It also possible to show that in fact we have the equality

$$\inf_{g: \|f - g\|_\infty \leq \frac{1}{2} c} TV (g, [a; b]) = TV^c (f, [a; b]).$$

For $f : [0; +\infty) \rightarrow \mathbb{R}$ we will denote

$$TV^c (f, [0; t]) =: TV^c (f, t).$$
Assymptotic properties of truncated variation

It appears that the truncated variation is closely related with $p-$variation, defined as

$$V^p (f, t) = \sup_n \sup_{0 \leq t_0 < t_1 < \ldots < t_n \leq t} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p.$$ 

Let

- $\mathcal{V}_p$ be the class of functions $f : [0; +\infty) \rightarrow \mathbb{R}$ with locally finite $p-$variation and
- $\mathcal{U}_p$ be the class of such functions $f$ that for any $t > 0$, 

$$\limsup_{c \downarrow 0} c^{p-1} TV^c (f, t) \in (0; +\infty)$$

In [5] it is shown that for any $p \geq 1$ and $\delta > 0$ we have inclusions $\mathcal{V}_p \subset \mathcal{U}_p \subset \mathcal{V}_{p+\delta}$ and for $p > 1$ these inclusions are strict.
Let \( X_t = B_t, \ t \geq 0, \) be a standard Brownian motion. Since it has infinite total variation on any interval \([0; t], \ t > 0,\) for any \( t > 0\)

\[
\lim_{c \downarrow 0} TV^c (X, t) = \infty.
\]

It may be of interest (due to the geometric interpretation of truncated variation and its asymptotic properties) to investigate the rate of \( TV^c (X, t)\) for small \( cs.\) The answer is the following
Limit distributions of truncated variation processes of Brownian motion with drift as $c \to 0$

Let $X_t = B_t$, $t \geq 0$, be a standard Brownian motion. Since it has infinite total variation on any interval $[0; t]$, $t > 0$, for any $t > 0$

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**Theorem ([4])**

For any $T > 0$ the process $c \cdot TV^c (X, t)$ converges almost surely in $(C[0; T], \mathbb{R})$ topology to the deterministic function $id : [0; T] \to \mathbb{R}, id(t) = t.$
Let now $X_t, t \geq 0$, be continuous a semimartingale, with the decomposition $X_t = X_0 + M_t + A_t$, with $M_t$ being a local martingale and $A_t$ being a continuous process with finite total variation.

Using the inequality $\max \{|x+y| - c, 0\} \leq \max \{|x| - c, 0\} + |y|$ we obtain, that

$$TV^c (X_t, t) \leq TV^c (M_t, t) + TV^0 (A_t, t)$$

and

$$TV^c (M_t, t) \leq TV^c (X_t, t) + TV^0 (A_t, t).$$

Hence we conclude easily that

$$\lim_{c \downarrow 0} c \cdot TV^c (X, t) = \lim_{c \downarrow 0} c \cdot TV^c (M, t)$$

whenever any of the above limits exists.
Generalisation for continuous semimartingales, cont.

We will use the Theorem from the previous slide, Dambis and Dubins-Schwarz Theorem saying that every continuous, local martingale $M_t, t \geq 0$, with $M_0 = 0$ and infinite total variation may be represented as

$$M_t = B_{\langle M, M \rangle_t},$$

where $B_t$ is a standard Brownian motion and the fact that the truncated variation does not depend on continuous and strictly increasing change of time variable. Utilizing above facts, we obtain that

$$\lim_{c \downarrow 0} c \cdot TV^c(X, t) = \lim_{c \downarrow 0} c \cdot TV^c(M, t) = \lim_{c \downarrow 0} c \cdot TV^c(B, \langle M, M \rangle_t) = \langle M, M \rangle_t.$$

Noticing that $\langle X, X \rangle_t = \langle M, M \rangle_t$, we finally obtain

$$\lim_{c \downarrow 0} c \cdot TV^c(X, t) = \langle X, X \rangle_t.$$
Concentration properties

Truncated variation seems to be more informative than $p$–variation:

- the finitness of $p$– variation may be recovered form the assymptotic properties of the truncated variation;
- for fixed $c > 0$, one may look at $TV^c$ as a random variable with the natural geometric interpretation mentioned.
Truncated variation seems to be more informative than $p$–variation:
- the finitness of $p$–variation may be recovered from the asymptotic properties of the truncated variation;
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It appears that for fixed $c > 0$ and for $X$ being a fBm or a diffusion with moderate growth, $TV^c$ reveals strong concentration properties. For example, for a standard Brownian motion we have

**Theorem ([1])**

*For the standard Brownian motion $X = B$, $t \cdot c^{-1}$ is comparable up to a universal constant with $E TV^c (X, t)$ and for some universal constants $A, B$ the Gaussian concentration holds*

\[ P(TV^c(B, t) \geq A \cdot t \cdot c^{-1} + B \sqrt{tu}) \leq \exp(-u^2), \quad \text{for} \quad u \geq 0. \]
The Skorohod problem

The truncated variation also appears to be related with the Skorohod problem on \([-c/2; c/2]\).
The truncated variation also appears to be related with the Skorohod problem on \([-c/2; c/2]\). Let
- \(D[0; +\infty)\) be a set of real-valued, càdlàg functions, defined on the interval \([0; +\infty)\),
- \(BV[0; +\infty)\) denote a subset of \(D[0; +\infty)\) consisting of functions with locally finite total variation and
- \(I[0; +\infty)\) denote a subset of \(D[0; +\infty)\) consisting of non-decreasing functions.
A pair of functions \((\phi, -\xi) \in D[0; +\infty) \times BV[0; +\infty)\) is said to be a solution of the \textbf{Skorohod problem} on \([-c/2, c/2]\) with starting condition \(\xi(0) = \xi^0\) for \(u \in D[0; +\infty)\) if the following conditions are satisfied:

1. for every \(t \geq 0\), \(\phi(t) = u(t) - \xi(t) \in [-c/2, c/2]\);
2. \(\xi = \xi_u - \xi_d\), where \(\xi_u, \xi_d \in L[0; +\infty)\) and the corresponding measures \(d\xi_u, d\xi_d\) are carried by \(\{t \geq 0 : \phi(t) = c/2\}\) and \(\{t \geq 0 : \phi(t) = -c/2\}\) respectively;
3. \(\xi(0) = \xi^0\).

For \(\xi^0 \in [u(0) - c/2; u(0) + c/2]\) the Skorohod problem has a unique solution.
Graphical interpretation of the Skorohod problem

Source: Pavel Krejčí, *Long-time behaviour of solutions to hyperbolic equations with hysteresis*, WIAS, Berlin,
Let $u^{c,\xi^0}$ be the solution of the Skorohod problem with starting condition $\xi^0 \in [u(0) - c/2; u(0) + c/2]$.

For any such $\xi^0$ and $t > 0$ we have

$$TV^c(u, t) \leq TV\left(u^{c,\xi^0}, t\right) \leq TV^c(u, t) + c.$$
Let $u^{c,\xi^0}$ be the solution of the Skorohod problem with starting condition
$\xi^0 \in [u(0) - c/2; u(0) + c/2]$. For any such $\xi^0$ and $t > 0$ we have

$$TV^c(u, t) \leq TV(u^{c,\xi^0}, t) \leq TV^c(u, t) + c.$$ 

For simplicity let us set $u^c = u^{c, u(0)}$. From the properties of the Skorohod map, we get

$$\int_0^t (u - u^c) du^c = \int_0^t \frac{c}{2} du^{c}_u - \int_0^t -\frac{c}{2} du^{c}_d = \frac{c}{2} \cdot TV(u^c, t),$$

where $u^{c}_u$ and $u^{c}_d$ are non-decreasing functions from the definition of the Skorohod problem, such that $u^c = u^{c, (0)} + u^{c}_u - u^{c}_d$. 
Since many years probabilists tried to define the stochastic integral in a pathwise way.
One of the earliest of such attempts is due to Wong and Zakai (1965). For $T > 0$ they considered the following approximation of Brownian paths:

(A) for all $t \in [0; T]$, $B^n_t \to B_t$ pointwise as $n \uparrow +\infty$, where $B^n$, $n = 1, 2, \ldots$, are continuous and have locally bounded variation;

(B) (A) and there exists such a locally bounded process $Z$ that for all $t \in [0; T]$, $|B^n_t| \leq Z_t$;

and stated the following
Wong-Zakai’s pathwise approach to the stochastic integral

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and stated the following theorem ([6])

Let $\psi(t, x)$ has continuous partial derivatives $\frac{\partial \psi}{\partial t}$ and $\frac{\partial \psi}{\partial x}$ and let $B^n$ satisfy (B), then for the Lebesgue-Stieltjes integrals $\int_0^T \psi(t, B^n_t) \, dB^n_t$, a.s.,

$$
\lim_{n \to \infty} \int_0^T \psi(t, B^n_t) \, dB^n_t = \int_0^T \psi(t, B_t) \, dB_t + \frac{1}{2} \int_0^T \frac{\partial \psi}{\partial x} (t, B_t) \, dt.
$$
Disadvantages of the Wong-Zakai construction

- The Wong-Zakai construction can not be extended when one considers different approximating sequences of the underlying Brownian motion in integrator and integrand.
- Using properties of the Skorohod map and the truncated variation it is relatively easy to construct an appropriate example.
Disadvantages of the Wong-Zakai construction

- The Wong-Zakai construction can not be extended when one considers different approximating sequences of the underlying Brownian motion in integrator and integrand.

- Using properties of the Skorohod map and the truncated variation it is relatively easy to construct an appropriate example.

Set: \( Y^n := B^{1/n^2} + n \left( B^{1/(2n^2)} - B^{1/n^2} \right) \). We easily check that \( Y^n \) and \( Z^n := B^{1/n^2} \) satisfy (A)-(B) for \( B \). We have \( B^{c/2} - B^c \geq c/4 \) on the set \( dB^c > 0 \) and \( B^{c/2} - B^c \leq -c/4 \) on the set \( dB^c < 0 \). Thus

\[
\int_0^1 Y^n \, dZ^n - \int_0^1 Z^n \, dZ^n = \int_0^1 n \left( B^{1/(2n^2)} - B^{1/n^2} \right) \, dB^{1/n^2} \\
\geq n \frac{1}{4n^2} \int_0^1 \left| dB^{1/n^2} \right| = \frac{n}{4} n^{-2} TV \left( B^{1/n^2}, 1 \right) \geq \frac{n}{4} n^{-2} TV^{1/n^2} \left( B, 1 \right).
\]
Let $X_t$, $t \geq 0$, be a process with càdlàg paths. The process $X^c$ obtained via the Skorohod map has the following properties:

(i) $X^c$ has locally finite total variation;
(ii) $X^c$ has càdlàg paths;
(iii) for every $T \geq 0$
\[ |X_t - X^c_t| \leq \frac{1}{2} c; \]
(iv) for every $T \geq 0$
\[ |\Delta X^c_t| \leq |\Delta X_t|, \]
where $\Delta X^c_t = X^c_t - X^c_{t-}$, $\Delta X_t = X_t - X_{t-}$;
(v) the process $X^c$ is adapted to the natural filtration of $X$. 
A generalisation of the Skorohod problem approximating sequence

In [3] for any càdlàg process $X$ the small generalisation is considered. For any $c > 0$ we consider a process $X^c$ such that

(i) $X^c$ has locally finite total variation;
(ii) $X^c$ has càdlàg paths;
(iii) for every $T \geq 0$ there exists such $K_T < +\infty$ that for every $t \in [0; T]$,
\[
|X_t - X^c_t| \leq K_T c;
\]
(iv) for every $T \geq 0$ there exists such $L_T < +\infty$ that for every $t \in [0; T]$,
\[
|\Delta X^c_t| \leq L_T |\Delta X_t|,
\]
where $\Delta X^c_t = X^c_t - X^c_{t-}$, $\Delta X_t = X_t - X_{t-}$;
(v) the process $X^c$ is adapted to the natural filtration of $X$. 
Further, in [3] the following theorems were proven.

**Theorem**

If processes $X$ and $Y$ are càdlàg semimartingales then for the sequence of the pathwise Lebesgue-Stieltjes integrals $\int_0^T Y_- dX^c$ we have

$$\int_0^T Y_- dX^c \rightarrow^P_{c\downarrow 0} \int_0^T Y_- dX + [X, Y]_{T}^{cont}.$$

$
\int_0^T Y_- dX$ denotes here the (semimartingale) stochastic integral and $[X, Y]_{T}^{cont}$ denotes here the continuous part of $[X, Y]$, i.e.

$$[X, Y]_{T}^{cont} = [X, Y]_{T} - \sum_{0<s\leq T} \Delta X_s \Delta Y_s.$$
Moreover

**Theorem**

*When* $c(n) > 0$ *and* $\sum_{n=1}^{+\infty} c(n)^2 < +\infty$ *then we have*

$$\int_0^T Y_-\,dX^{c(n)} \to \int_0^T Y_-\,dX + [X, Y]_{T}^{cont} \text{ a.s.}$$
Drawbacks of the construction presented

Unfortunately, the construction presented does not work for any càglàd integrand $Y$.

It is possible to construct a **continuous, globally bounded, adapted to the natural Brownian filtration** process $Y$ and a sequence $B^{c(n)}$, $n = 1, 2, \ldots$, satisfying all conditions (i)-(v) for $X = B$ such that the integral

$$\int_0^1 Y \, dB^{c(n)}$$

diverges.
First (cf. [3]) we define sequence $b(n)$, $n = 1, 2, \ldots$ in the following way $b(1) = 1$ and for $n = 2, 3, \ldots$}

$$b(n) = n^2 b(n - 1)^6.$$ 

Now we define $a(n) := b(n)^{1/2}$, $c(n) := b(n)^{-1}$ and set

$$Y := \sum_{n=2}^{\infty} a(n) \left( B - B^{c(n)} \right).$$

The proof also utilises the concentration properties of $TV^c$. 
Bichteler’s construction

The remarkable Bichteler’s approach provides pathwise construction for integration of any adapted càdlàg process \( Y \) with càdlàg semimartingale integrator \( X \) and is based on the approximation

\[
\lim_{n \to \infty} \sup_{0 \leq t \leq T} \left| Y_0X_0 + \sum_{i=1}^{\infty} Y_{\tau_i^n \wedge t} \left( X_{\tau_i^n \wedge t} - X_{\tau_i^n \wedge t-1} \right) - \int_0^t Y_- dX \right| = 0 \text{ a.s.},
\]

Remark

In fact, given \( c(n) > 0 \),

\[
\sum_{n=1}^{\infty} c^2(n) < +\infty,
\]

Bichteler’s construction works for any sequence \( \tau_n = (\tau_n^i) \), \( i = 0, 1, 2, \ldots \), of stopping times, such that \( \tau_0^0 = 0 \) and for \( i = 1, 2, \ldots \),

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\tau_n^i = \inf \{ s > \tau_{n-1}^i : |Y_s - Y_{\tau_{n-1}^i}| \geq c(n) \}. 
\]
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where $\tau^n = (\tau_i^n), i = 0, 1, 2, \ldots$, is the following sequence of stopping times: $\tau_0^n = 0$ and for $i = 1, 2, \ldots$,

$$\tau_i^n = \inf \left\{ s > \tau_{i-1}^n : \left| Y_s - Y_{\tau_{i-1}^n} \right| \geq 2^{-n} \right\}.$$
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Remark

*In fact, given $c(n) > 0$, $\sum_{n=1}^{\infty} c^2(n) < +\infty$, Bichteler’s construction works for any sequence $\tau^n = (\tau^n_i)$, $i = 0, 1, 2, \ldots$, of stopping times, such that $\tau^n_0 = 0$ and for $i = 1, 2, \ldots$, $\tau^n_i = \inf \left\{ s > \tau^n_{i-1} : \left| Y_s - Y_{\tau^n_{i-1}} \right| \geq c(n) \right\}$.***
Some references


Thank you!