Subjective probability and arbitrage
Where are we?

- Choice under risk - probabilities given objectively

  - Expected Utility Theory
    - crucial axioms:
      - Transitivity - giving order/ranking among alternatives
      - Independence - giving cardinality
    - Stochastic dominance (partial orders)
      - FOSD is equivalent to EU with increasing utility (monotonicity)
      - SOSD is equivalent to EU with concave utility (risk aversion)

  - Risk attitudes
    - risk averse - concave utility
    - risk neutral - linear utility
    - risk seeking - convex utility
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Risk aversion (concave utility) can be measured **locally**:

- **Absolute Risk Aversion**

$$ARA(x) = -\frac{u''(x)}{u'(x)}$$
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Based on these measures we define classes of utility functions:

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  \item **Constant Absolute Risk Aversion (CARA)** - wealth invariant
  \item **Decreasing Absolute Risk Aversion (DARA)** - wealthier - accept more
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  \item **Logarithmic utility function**
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CARA utility functions

\[ U_a(x) = \frac{1}{a} - \frac{1}{a} \exp(-ax) \]

CRRA utility functions

\[ U_a(x) = \frac{1}{1-a} \cdot x^{1-a} - 1/(1-a) \]

Red line is log
Two issues

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- Why don’t we give up some of the axioms and invent the new ones?
  - Non-expected utility models and behavioral economics - later
Games

a) The dealer will pay $1 if you flip a coin and it lands head up. How much will you pay to play this game?
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b) The dealer will pay $2 if you roll a die and it lands with a 6 up. How much will you pay to play this game?
c) The dealer will pay $2 if the card you draw has a rank at least as high as the rank of the card he draws. How much will you pay to play this game?
Bruno de Finetti definition of probability

Let us suppose that an individual is obliged to evaluate the rate $p$ at which he would be willing to exchange the possession of an arbitrary sum $S$ (positive or negative) contingent on the occurrence of a given event, $E$, for the possession of the sum $pS$;
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From symmetry to probability

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- Hence we would pay the same if a head were replaced by a tail.

\[ p + \frac{1}{2} = 1 \implies p = \frac{1}{2} \]
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- We believe that there is either tail or head: \( p + p = 1 \) implies \( p = 0.5 \) and we would pay up to $0.5.

- **Game b)**: The same with a die:
- \( p + p + p + p + p + p = 1 \) and hence \( p = \frac{1}{6} \) and we would pay up to $0.33.
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- There are $52 \times 51$ possibilities (probability of each is $\frac{1}{52 \times 51}$)
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Once the dealer draws a card, there is 3 more cards in the deck with the same values, hence there is $52 \times 3$ outcomes in which the two cards have the same value.
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- Of the remaining $52 \cdot 51 - 51 \cdot 3 = 52 \cdot 48$ outcomes, the player wins half, $52 \cdot 24$. 

So the probability of winning is $\frac{52 \cdot 3 + 52 \cdot 24}{52 \cdot 51} = \frac{27}{51}$. Hence we would pay up to $\$1/period.math$03846 for this game.
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Vocabulary of gambling

- **Total stake** $S$: the amount that the player can win
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- **Dealer’s stake** $(1 - p)S$: the amount that the dealer is putting up
- **Player’s odds** $\frac{p}{1 - p}$, often phrased "$p/(1 - p)$ to 1"
Definition of subjective probability:

- Let $A \subseteq \Omega$. 

Consider the following bet which pays $1 if there was life on Mars 1 billion years ago ($A$), $0 if there was not, and tomorrow the answer will be revealed.

Your opponent will be able to choose either to buy such a promise from you at the price you have set, or require you to buy such a promise from your opponent, still at the same price.

In other words: you set the odds, but your opponent decides which side of the bet will be yours.

The price you set is the "operational subjective probability" that you assign to the proposition on which you are betting.
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The set of all events will be denoted by $\mathcal{E}$. The event that is certain to occur is $\Omega$. The event that will never occur is $\emptyset$. Definition of subjective conditional probability: Let $A, B \subseteq \Omega$. Consider the following bet which pays $1 if there was life on Mars 1 billion years ago ($A$) and there is life on Mars now ($B$), $0 if there was no life on Mars 1 billion years ago, but there is now. If there is no life on Mars now and the answer will be revealed tomorrow. Then the conditional probability of $A$ given $B$, written $P(A \mid B)$, is the price which you are willing to exchange this bet for.
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Definition of subjective conditional probability:

Let $A$ and $B \subseteq \Omega$.

Consider the following bet which pays $1 if there was life on Mars 1 billion years ago ($A$) and there is life on Mars now ($B$), $0 if there was no life on Mars 1 billion years ago, but there is now, is called off if there is no life on Mars now, and the answer will be revealed tomorrow.

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Dutch book

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The above is exactly equivalent to Subjective Expected Utility.
Axioms of probability

a) \( P(A) \geq 0, \forall A \in \mathcal{E} \)
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P(A \cup B) = P(A) + P(B)
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$$P(C_1 \cup C_2 \cup \ldots \cup C_n) = P(C_1) + P(C_2 \cup \ldots \cup C_n)$$
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  \( P(C_1 \cup C_2 \cup \ldots \cup C_n) = P(C_1) + P(C_2 \cup \ldots \cup C_n) \)
- Do the same many times and you get
  \( P(C_1 \cup C_2 \cup \ldots \cup C_n) = P(C_1) + P(C_2) + \ldots + P(C_n) \)
Axiom a) Possibility: $P(A) \geq 0$

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Axiom a) Possibility: $P(A) \geq 0$

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- Suppose that $p < 0$
Axiom a) Possibility: \( P(A) \geq 0 \)

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FREE LUNCH!!!
Arbitrage and independence

▶ Violation of independence
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\[ \text{̄} \text{A} \text{s} \text{s} \text{u} \text{m} \text{e} \text{m} \text{e} \text{t} \text{t} \text{i} \text{t} \text{i} \text{o} \text{n} \text{a} \text{t} \text{i} \text{t} \text{y} \text{ } \text{t} \text{h} \text{a} \text{t} \text{ } \text{e} \text{v} \text{e} \text{n} \text{t} \text{ } \text{E} \text{ } \text{o} \text{c} \text{c} \text{u} \text{r} \text{s} \text{ } \text{w} \text{i} \text{t} \text{h} \text{ } \text{p} \text{r} \text{o} \text{b} \text{a} \text{b} \text{i} \text{l} \text{i} \text{t} \text{i} \text{y} \text{ } \alpha. \]
Arbitrage and independence

▶ Violation of independence
  ▶ Mr. X has the following preferences: $L_1 > L_2$ and
    $\langle L_1, \alpha; L_3, 1 - \alpha \rangle < \langle L_2, \alpha; L_3, 1 - \alpha \rangle$.
  ▶ They violate IIR.

\(^1\text{Assume it is common knowledge that event } E \text{ occurs with probability } \alpha.\)
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  - I propose him the following deal:
    1. Take $(L_1, \alpha; L_3, 1 - \alpha)$
    2. Exchange for $(L_2, \alpha; L_3, 1 - \alpha)$ and pay me $\epsilon_1 > 0$
    3. Agree that when event $E$ occurs, exchange $L_2$ with $L_1$ and pay me $\epsilon_2 > 0$
    4. Repeat from step 2

---

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FREE LUNCH!!!
We have shown that

- if you violate **Expected Utility** (subjective or objective) axioms,
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- then you are vulnerable to a Dutch book

The converse statement is also true, but one must be careful as to how we interpret expected utility.
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What we miss is the converse statement:

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The converse statement is also true, but one must be careful as to how we interpret expected utility.
Dual independence and arbitrage

Let $x \equiv (x, p)$ and $y \equiv (y, q)$ be two lotteries.
Dual independence and arbitrage

- Let $x \equiv (x, p)$ and $y \equiv (y, q)$ be two lotteries.
- Two ways of mixing two lotteries.
Dual independence and arbitrage

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Yaari (1985) - DB no.4: Suppose that an agent exhibits the following preference pattern:

\[
x \succ (y, q) \text{ and } y \succ \frac{x}{2} \succ \frac{y}{2}
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Dual independence and arbitrage

Let $x \equiv (x, p)$ and $y \equiv (y, q)$ be two lotteries.

Two ways of mixing two lotteries.

- with probability $\alpha$ play lottery $x$ and with probability $1 - \alpha$ play lottery $y$, denoted by $\alpha x \oplus (1 - \alpha)y$.
- an agent owns $\alpha$ shares of lottery $x$ and $1 - \alpha$ shares of lottery $y$ denoted by $\alpha x + (1 - \alpha)y$. 

Yaari (1985) - DB no.4: suppose that an agent exhibits the following preference pattern:

$$(x, p) \succ (y, q) \text{ and } (y, 2q) \succ (x, 2p) \quad (1)$$
Dual independence and arbitrage

- Let \( x \equiv (x, p) \) and \( y \equiv (y, q) \) be two lotteries.
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    - Notice that we need the joint distribution of \( x \) and \( y \) apart from the marginals.

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(x, p) \succ (y, q) \quad \text{and} \quad \left(\frac{y}{2}, q\right) \succ \left(\frac{x}{2}, p\right)
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The above pattern is in violation with the dual independence axiom.
The above pattern is in violation with the **dual independence axiom**.

\[ x > y \text{ and } \alpha x + (1 - \alpha)z > \alpha y + (1 - \alpha)z, \quad \forall \alpha \in (0, 1) \]
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Note that this preference pattern is possible under expected utility.
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**Is it really so?**