Preference relation, choice rule and utility function
• **Individual decision making**
  – **under Certainty**
  • Revealed preference and **ordinal** utility theory

If \( u() \) is a utility function, then any strictly increasing transformation \( g \circ u() \) is a utility function representing the same preferences.

---

• **Individual decision making**
  – **under Certainty**
  • Revealed preference and utility theory
• Individual decision making
  – under Certainty
  • Choice functions

Weak axiom of revealed preference (WARP)

NOT ALLOWED

You go to a restaurant in while you are on vacation in Tuscany and you are given the following menu:
• bistecca
• pollo

The cook announces that he can also serve
• trippa alla fiorentina
Preference relations

• Mathematically – binary relations in the set of decision alternatives:
  – $X$ – decision alternatives
  – $X^2$ – all pairs of decision alternatives
  – $R \subseteq X^2$ – binary relation in $X$, selected subset of ordered pairs of elements of $X$
  – if $x$ is in relation $R$ with $y$, then we write $xRy$ or $(x,y) \in R$

• Examples of relations:
  – ”Being a parent of” is a binary relation on a set of human beings
  – ”Being a hat” is a binary relation on a set of objects
  – ”$x+y=z$” is 3-ary relation on the set of numbers
  – ”$x$ is better than $y$ more than $x$’ is better than $y$’” is a 4-ary relation on the set of alternatives.
Logical preliminary

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>~p</th>
<th>~q</th>
<th>p \Rightarrow q &lt;=&gt; ~p \lor q &lt;=&gt; ~q \Rightarrow ~p</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

\[ p \lor q \equiv (p \land q) \land \neg p \land \neg q \]

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>~p</th>
<th>~q</th>
<th>p \lor q</th>
<th>~q \land ~p</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Binary relations – basic properties

- complete: \( xRy \) or \( yRx \)
- reflexive: \( xRx \) (\( \forall x \))
- irreflexive: not \( xRx \) (\( \forall x \))
- transitive: if \( xRy \) and \( yRz \), then \( xRz \)
- symmetric: if \( xRy \), then \( yRx \)
- asymmetric: if \( xRy \), then not \( yRx \)
- antisymmetric: if \( xRy \) and \( yRx \), then \( x=y \)
- negatively transitive: if not \( xRy \) and not \( yRz \), then not \( xRz \)
  - equivalent to: \( xRz \) implies \( xRy \) or \( yRz \)
- acyclic: if \( x_1Rx_2, x_2Rx_3, \ldots, x_{n-1}Rx_n \) imply \( x_1 \neq x_n \)
Exercise – check the properties of the following relations

- \( R_1 \): (among people), to have the same colour of the eyes
- \( R_2 \): (among people), to know each other
- \( R_3 \): (in the family), to be an ancestor of
- \( R_4 \): (among real numbers), not to have the same value
- \( R_5 \): (among words in English), to be a synonym
- \( R_6 \): (among countries), to be at least as good in a rank-table of summer olympics

<table>
<thead>
<tr>
<th></th>
<th>( R_1 )</th>
<th>( R_2 )</th>
<th>( R_3 )</th>
<th>( R_4 )</th>
<th>( R_5 )</th>
<th>( R_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>complete</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>reflexive</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>irreflexive</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>transitive</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>symmetric</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>asymmetric</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>negatively transitive</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2 questionnaires

\( P \) (for all distinct \( x \) and \( y \) in \( X \)):
How do you compare \( x \) and \( y \)? Tick one and only one of the following three options:
- I prefer \( x \) to \( y \) (this answer is denoted as \( x \succ y \) or \( xP y \)).
- I prefer \( y \) to \( x \) (this answer is denoted by \( y \succ x \), or \( y P x \)).
- Neither of the first two. I am indifferent (this answer is denoted by \( x \sim y \) or \( x I y \)).

\( R \) (for all \( x, y \in X \), not necessarily distinct):
Is \( x \) at least as preferred as \( y \)? Tick one and only one of the following two options:
- Yes
- No
2 questionnaires

We exclude right away:

- A lack of ability to compare (I have no opinion, they are incomparable)
- A dependence on other factors (depends on what my parents think)
- Intensity of preferences (I somewhat prefer x, I love x and hate y)

Rational preference relation

- P is a (rational) strict preference relation in X, if it is:
  - asymmetric
  - negatively transitive
  - acyclic
  - transitive
  - ...

- Q is a (rational) weak preference relation in X, if it is:
  - complete
  - transitive
  - acyclic
  - reflexive
  - ...

- Completeness implies reflexivity (be sure that you understand)
- Asymmetry + negative transititivity implies transitivity (prove)
- Etc.
Relationship between strict and weak preferences

Let $R$ be a weak preference relation (transitive, complete)
- $R$ generates strict preference relation – $P$:
  - $xPy$, iff $xRy$ and not $yRx$
- $R$ generates indifference relation – $I$:
  - $xIy$, iff $xRy$ and $yRx$

Let $P$ be a strict preference relation (asymmetric and negatively transitive)
- $P$ generates weak preference relation – $R$:
  - $xRy$, iff not $y Px$
- $P$ generates indifference relation – $I$:
  - $xIy$, iff not $xPy$ and not $yPx$

An example $[X=\{a,b,c,d\}]$

- $R=\{(a,a), (a,b), (a,c), (a,d), (b,a), (b,b), (b,c), (b,d), (c,c), (c,d), (d,d)\}$ generates:
  - $P=\{(a,c), (a,d), (b,c), (b,d), (c,d)\}$
  - $I=\{(a,a), (a,b), (a,c), (b,b), (c,c), (d,d)\}$

And the other way around:
- $P=\{(a,c), (a,d), (b,c), (b,d), (c,d)\}$ generates:
  - $R=\{(a,a), (a,b), (a,c), (a,d), (b,a), (b,b), (b,c), (b,d), (c,c), (c,d), (d,d)\}$
  - $I=\{(a,a), (a,b), (b,b), (c,c), (d,d)\}$
- Observe that $R=P \cup I$
Exercise

- $X = \{a, b, c, d\}$
- $P = \{(a, d), (c, d), (a, b), (c, b)\}$

- Find $R$ and $I$

---

$P$ vs $R$ ($xPy \iff xRy \land \sim yRx$)

- $R$ is complete iff $P$ is asymmetric

- $R$ is transitive iff $P$ is negatively transitive
Properties of I

- I is an equivalence relation iff it is:
  - reflexive
  - transitive
  - symmetric

- Can we start with I as a primitive?
  - reflexive
  - symmetric
  - transitive
- No – we wouldn’t be able to order the abstraction classes

---

Proof of the properties of I from the properties of R (xly ⇔ xRy ∧ yRx)

- reflexive (xlx)
  - obvious – using reflexivity of R we get xRx

- transitive (xly ∧ ylz ⇒ xlz)
  - predecessor means that xRy ∧ yRx ∧ yRz ∧ zRy
  - using transitivity we get xRz ∧ zRx, QED

- symmetric (xly ⇒ ylx)
  - predecessor means that xRy ∧ yRx, QED
Another definition of rational preferences

• Is it enough to use a relation P that is:
  – asymmetric
  – acyclic (not necessarily negatively transitive)

• No – let’s see an example

P from the previous slide – an example

• Mr X got ill and for years to come will have to take pills twice a day in an interval of exactly 12 hours. He can choose the time however.

• All the decision alternatives are represented by a circle with a circumference 12 (a clock). Let’s denote the alternatives by the length of an arc from a given point (midnight/noon).

• Mr X has very peculiar preferences – he prefers y to x, if y=x+π, otherwise he doesn’t care

• Thus yPx, if y lies on the circle π units farther (clockwise) than x
Exercise

• What properties does P have?
  – Asymmetry (YES)
  – negative transitivity (NO)
  – Transitivity (NO)
  – Acyclicity (YES)

• P generates „weird” preferences:
  – Not transitive
    • 1+2π better than 1+π,
    • 1+π better than 1,
    • 1+2π equally good as 1
  – Not negatively transitive
    • 1 equally good as 1+π/2,
    • 1+π/2 equally good as 1+π,
    • 1 worse than 1+π

Another definition of rational preferences

• What if we take P?
  – asymmetric
  – transitive (not necessarily negatively transitive)
  – thus acyclic

• First let’s try to find an example
• Then let’s think about such preferences
Asymmetric, transitive, not negatively transitive relation – intuition

- $X = \{R_+\}$, $xPy \iff x > y + 5$ (I want more, but I am insensitive to small changes)

- Properties of P:
  - asymmetric – obviously
  - transitive – obviously
  - negatively transitive?
    - 11 P 5, but
      - neither 11 P 8, nor 8 P 5

- Thus I is not transitive: 11 I 8 and 8 I 5, but not 11 I 5
Properties of preferences – a summary

- **P („better than“)** – asymmetric, negatively transitive
- **P („better than“)** – asymmetric, transitive
- **P („better than“)** – asymmetric, acyclic
- **R („at least as good as“)** – transitive, complete

**Colours, insensitivity to small changes**

**eg. Mr X**

Choice functions – a formal definition

- **Notation:**
  - \( X \) set of decision alternatives
  - \( B \subseteq 2^X, \emptyset \notin B \) available menus (non-empty subsets of \( X \))
  - \( C : B \to B \) choice function, working for every menu

- **(Technical) properties:**
  - \( C(B) \neq \emptyset \) always a choice
  - \( C(B) \subseteq B \) out of a menu

- If \( C(B) \) contains a single element \( \Rightarrow \) this is the choice
- If more elements \( \Rightarrow \) these are possible choices (not simultaneously, the decision maker picks one in the way which is not described here)
Preferences once more (this time strict)

- Let $X$ represent some set of objects
- Often in economics $X \subseteq \mathbb{R}^K$ is a space of consumption bundles
  - E.g. 3 commodities: beer, wine and whisky
  - $x = (x_1, x_2, x_3)$ ($x_1$ cans of beer, $x_2$ bottles of wine, $x_3$ shots of whisky)
- We present the consumer pairs $x$ and $y$ and ask how they compare
- Answer $x$ is better than $y$ is written $x \succ y$ and is read $x$ is strictly preferred to $y$
- For each pair $x$ and $y$ there are 4 possible answers:
  - $x$ is better than $y$, but not the reverse
  - $y$ is better than $x$, but not the reverse
  - neither seems better to her
  - $x$ is better than $y$, and $y$ is better than $x$
Assumptions on strict preferences

- We would like to exclude the fourth possibility right away

**Assumption 1:** Preferences are asymmetric. There is no pair $x$ and $y$ from $X$ such that $x \succ y$ and $y \succ x$

- Possible objections:
  - What if decisions are made in different time periods?
    - change of tastes
    - addictive behavior (1 cigarette $\succ$ 0 cigarettes $\succ$ 20 cigarettes changed to 20 cigarettes $\succ$ 1 cigarette $\succ$ 0 cigarettes)
    - dual-self model
  - Dependence on framing
    - E.g. Asian disease
Assumptions on strict preferences

**Assumption 2:** Preferences are **negatively transitive**: If \( x > y \), then for any third element \( z \), either \( x > z \), or \( z > y \), or both.

- **Possible objections:**
  - Suppose objects in \( X \) are bundles of cans of beer and bottles of wine \( x = (x_1, x_2) \)
  - No problem comparing \( x = (21, 9) \) with \( y = (20, 8) \)
  - Suppose \( z = (40, 2) \). Negative transitivity demands that either \( (21, 9) > (40, 2) \), or \( (40, 2) > (20, 8) \), or both.
  - The consumer may say that comparing \( (40, 2) \) with either \( (20, 8) \) or \( (21, 9) \) is too hard.
  - Negative transitivity rules this out.
Weak preferences and indifference induced from strict preferences

- Suppose our consumer’s preferences are given by the relation $>$. 

**Definition:** For $x$ and $y$ in $X$,

- write $x \succeq y$, read "$x$ is weakly preferred to $y$", if it is not the case that $y > x$.
- write $x \sim y$, read "$x$ is indifferent to $y$", if it is not the case that either $x > y$ or $y > x$.

- Problem with noncomparability: if the consumer is unable to compare $(40, 2)$ with either $(20, 8)$ or $(21, 9)$, it doesn’t mean she is indifferent between them.
Proposition: If $\succ$ is asymmetric and negatively transitive, then:

- weak preference $\succeq$ is complete and transitive
- indifference $\sim$ is reflexive, symmetric and transitive
- Additionally, if $w \sim x, x \succ y,$ and $y \sim z,$ then $w \succ y$ and $x \succ z.$

The first two were proved previously. The third may be proved at home.
Proposition: If $\succ$ is asymmetric and negatively transitive, then $\succ$ is irreflexive, transitive and acyclic.

Proof.

- Irreflexive by asymmetry
- Transitivity:
  - Suppose that $x \succ y$ and $y \succ z$
  - By negative transitivity and $x \succ y$, either $x \succ z$ or $z \succ y$
  - Since $y \succ z$, asymmetry forbids $z \succ y$. Hence $x \succ z$
- Acyclicity:
  - If $x_1 \succ x_2, x_2 \succ x_3, \ldots, x_{n-1} \succ x_n$, then transitivity implies $x_1 \succ x_n$
  - Asymmetry (or irreflexivity) implies $x_1 \neq x_n$

Quod Erat Demonstrandum (QED)
Properties of preferences – a summary

P ("better than") – asymmetric, negatively transitive

R ("at least as good as") – transitive, complete

colours, insensitivity to small changes

e.g. Mr X

P ("better than") – asymmetric, transitive

P ("better than") – asymmetric, acyclic

Choice functions – a formal definition

• Notation:
  
  \[
  X \quad \text{set of decision alternatives}
  \]

  \[
  B \subset 2^X, \emptyset \not\in B \quad \text{available menus (non-empty subsets of } X)\]

  \[
  C : B \rightarrow B \quad \text{choice function, working for every menu}\]

• (Technical) properties:

  \[
  C(B) \neq \emptyset \quad \text{always a choice}\]

  \[
  C(B) \subset B \quad \text{out of a menu}\]

• If \( C(B) \) contains a single element \( \rightarrow \) this is the choice

• If more elements \( \rightarrow \) these are possible choices (not simultaneously, the decision maker picks one in the way which is not described here)
An exercise

- Let $X=\{a,b,c\}$, $\mathcal{P}(X)=2^X$

- Write down the following choice functions:
  - $C_1$: always $a$ (if possible), if not – it doesn’t matter
  - $C_2$: always the first one in the alphabetical order
  - $C_3$: whatever but not the last one in the alphabetical order (unless there is just one alternative available)
  - $C_4$: second first alphabetically (unless there is just one alternative)
  - $C_5$: disregard $c$ (if technically it is possible), and if you do disregard $c$, also disregard $b$ (if technically possible)

The solution

<table>
<thead>
<tr>
<th>B</th>
<th>$C_1(B)$</th>
<th>$C_2(B)$</th>
<th>$C_3(B)$</th>
<th>$C_4(B)$</th>
<th>$C_5(B)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>{a}</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{b}</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{c}</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{a,b}</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{a,c}</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{b,c}</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{a,b,c}</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Choice rule induced by preference relation

- How do we relate preference relation with choice behavior?

**Definition:** Given a preference relation $\succ$ on a set of objects $X$ and a nonempty subset $A$ of $X$, the set of acceptable alternatives from $A$ according to $\succ$ is defined to be:

$$c(A; \succ) = \{x \in A : \text{There is no } y \in A \text{ such that } y \succ x\}$$

Several things to note:

- $c(A; \succ)$ by definition subset of $A$
- $c(A; \succ)$ may contain more than one element (anything will do)
Properties of such choice rule

- In some cases, $c(A; >)$ may contain no elements at all
  - $X = [0, \infty)$ with $x \in X$ representing $x$ dollars
  - $A \subseteq X, A = \{1, 2, 3, \ldots\}$
  - Always prefers more money to less $x > y$ whenever $x > y$
  - Then $c(A; >)$ will be empty
  - The same when $A = [0, 10)$ and money is infinitely divisible

- In the examples above, $c(A; >)$ is empty because $A$ is too large or not nice - it may be that $c(A; >)$ is empty because $>$ is badly behaved
  - Suppose $X = \{x, y, z, w\}$, and $x > y, y > z, \text{and } z > x$. Then $c(\{x, y, z\}; >) = \emptyset$
**WARP**

- **Weak Axiom of Revealed Preference**: if \( x \) and \( y \) are both in \( A \) and \( B \) and if \( x \in c(A) \) and \( y \in c(B) \), then \( x \in c(B) \) (and \( y \in c(A) \)).

- It may be decomposed into two properties:
  - **Sen’s property** \( \alpha \): If \( x \in B \subseteq A \) and \( x \in c(A) \), then \( x \in c(B) \).
  - **Sen’s property** \( \beta \): If \( x, y \in c(A), A \subseteq B \) and \( y \in c(B) \), then \( x \in c(B) \).

- If the world champion in some game is a Pakistani, then he must also be the champion of Pakistan.

- **Observation**:
  - Property \( \alpha \) specializes to the case \( A \subseteq B \)
  - Property \( \beta \) specializes to the case \( B \subseteq A \).
Rational preferences induce rational choice rule

**Proposition:** Suppose that \( > \) is asymmetric and negatively transitive. Then:

(a) For every finite set \( A \), \( c(A; >) \) is nonempty

(b) \( c(A; >) \) satisfies WARP

**Proof.**

**Part I: \( c(A; >) \) is nonempty:**

- We need to show that the set \( \{ x \in A : \forall y \in A, y \not> x \} \) is nonempty
- Suppose it was empty - then for each \( x > A \) there exists a \( y \in A \) such that \( y > x \).
- Pick \( x_1 \in A \) (\( A \) is nonempty), and let \( x_2 \) be \( x_1 \)'s "\( y \)".
- Let \( x_3 \) be \( x_2 \)'s "\( y \)", and so on. In other words, take \( x_1, x_2, x_3 \ldots \in A \), such that \( \ldots x_n > x_{n-1} > \ldots > x_2 > x_1 \)
- Since \( A \) is finite, there must exist some \( m \) and \( n \) such that \( x_m = x_n \) and \( m > n \).
- But this would be a cycle. Contradiction.
- So \( c(A; >) \) is nonempty. **End of part I.**
Part II: \( c(A, \succ) \) satisfies WARP:

- Suppose \( x \) and \( y \) are in \( A \cap B \), \( x \in c(A, \succ) \) and \( y \in c(B, \succ) \).
- Since \( x \in c(A, \succ) \) and \( y \in A \), we have that \( y \nsubsetneq x \).
- Since \( y \in c(B, \succ) \), we have that for all \( z \in B \), \( z \nsubsetneq y \).
- By negative transitivity of \( \succ \), for all \( z \in B \) it follows that \( z \nsubsetneq x \).
- This implies \( x \in c(B, \succ) \).
- Similarly for \( y \in c(A, \succ) \). End of part II.

QED
Choice rules as a primitive

- Let us now reverse the process: We observe choice and want to deduce preferences.

**Definition:** A choice function on $X$ is a function $c$ whose domain is the set of all nonempty subsets of $X$, whose range is the set of all subsets of $X$, and that satisfies $c(A) \subseteq A$, for all $A \in X$.

- **Assumption:** The choice function $c$ is nonempty valued: $c(A) \neq \emptyset$, for all $A$.

- **Assumption:** The choice function $c$ satisfies Weak Axiom of Revealed Preference: If $x, y \in A \cap B$ and if $x \in c(A)$ and $y \in c(B)$, then $x \in c(B)$ and $y \in c(A)$. 

Proposition: If a choice function $c$ is nonempty valued and satisfies property $\alpha$ and property $\beta$ (and hence WARP), then there exists a preference relation $\succ$ such that $c$ is $c(\cdot, \succ)$.
Rational choice rule induces rational preferences

Proof.

- Define \( \succ \) as follows:

\[
x \succ y \iff x \not\equiv y \text{ and } c(\{x, y\}) = \{x\}
\]

- This relation is obviously asymmetric.

Part I: \( \succ \) is negatively transitive

- Suppose that \( x \not\succ y \) and \( y \not\succ z \), but \( x \succ z \).
- \( x \succ z \) implies that \( \{x\} = c(\{x, z\}) \), thus \( z \not\in c(\{x, y, z\}) \) by property \( \alpha \)
- Since \( z \in c(\{y, z\}) \), this implies \( y \not\in c(\{x, y, z\}) \) again by property \( \alpha \)
- Since \( y \in c(\{x, y\}) \), implies \( x \not\in c(\{x, y, z\}) \) again by...
- Which is not possible since \( c \) is nonempty valued. Contradiction
- Hence \( \succ \) is negatively transitive. End of part I.
Rational choice rule induces rational preferences

Part II: $c(A, \succ) = c(A)$ for all sets $A$

- Fix a set $A$
  
  (a) If $x \in c(A)$, then for all $z \in A$, $z \not\succ x$. For if $z \succ x$, then $c(\{x, z\}) = \{z\}$, contradicting property $\alpha$. Thus $x \in c(A, \succ)$
  
  (b) If $x \notin c(A)$, then let $z$ be chosen arbitrarily from $c(A)$. We claim that $c(\{z, x\}) = \{z\}$ - otherwise property $\beta$ would be violated. Thus $z \succ x$ and $x \notin c(A, \succ)$.

- Combining (a) and (b), $c(A, \succ) = c(A)$ for all $A$. End of part II.

QED
Utility representation

**Definition:** Function $u : X \rightarrow \mathbb{R}$ represents rational preference relation $\succ$ if for all $x, y \in X$ the following holds

$$x \succ y \iff u(x) > u(y)$$

- The representation is always well defined since $\succ$ on $\mathbb{R}$ satisfies negative transitivity and asymmetry.

**Proposition:** If $u$ represents $\succ$, then for any strictly increasing function $f : \mathbb{R} \rightarrow \mathbb{R}$, the function $v(x) = f(u(x))$ represents $\succ$ as well. **Proof.**

$$x \succ y$$
$$u(x) > u(y)$$
$$f(u(x)) > f(u(y))$$
$$v(x) > v(y)$$

QED
Lemma:
In any finite set $A \subseteq X$, there is a minimal element (similarly, there is also a maximal element).

Proof:
By induction on the size of $A$. If $A$ is a singleton, then by completeness its only element is minimal. For the inductive step, let $A$ be of cardinality $n + 1$ and let $x \in A$. The set $A - \{x\}$ is of cardinality $n$ and by the inductive assumption has a minimal element denoted by $y$. If $x \preceq y$, then $y$ is minimal in $A$. If $y \succeq x$, then by transitivity $z \succeq x$ for all $z \in A - \{x\}$, and thus $x$ is minimal.
Utility representation for finite sets

Claim:
If $\succeq$ is a preference relation on a finite set $X$, then $\succeq$ has a utility representation with values being natural numbers.

Proof:
We will construct a sequence of sets inductively. Let $X_1$ be the subset of elements that are minimal in $X$. By the above lemma, $X_1$ is not empty. Assume we have constructed the sets $X_1, \ldots, X_k$. If $X = X_1 \cup X_2 \cup \cdots \cup X_k$, we are done. If not, define $X_{k+1}$ to be the set of minimal elements in $X - X_1 - X_2 - \cdots - X_k$. By the lemma $X_{k+1} \neq \emptyset$. Because $X$ is finite, we must be done after at most $|X|$ steps. Define $U(x) = k$ if $x \in X_k$. Thus, $U(x)$ is the step number at which $x$ is “eliminated”. To verify that $U$ represents $\succeq$, let $a \succ b$. Then $a \notin X_1 \cup X_2 \cup \cdots X_{U(b)}$ and thus $U(a) > U(b)$. If $a \sim b$, then clearly $U(a) = U(b)$.
**Definition:** A preference relation $\succ$ on $X$ is continuous if for all $x, y \in X$, $x \succ y$ implies that there is an $\epsilon > 0$ such that $x' \succ y'$ for any $x'$ and $y'$ such that $d(x, x') < \epsilon$ and $d(y, y') < \epsilon$.

**Proposition:** Assume that $X$ is a convex subset of $\mathbb{R}^n$. If $\succ$ is a continuous preference relation on $X$, then $\succ$ has a continuous utility representation.
Utility representation result II

Monotonicity:
The relation $\succeq$ satisfies *monotonicity at the bundle* $y$ if for all $x \in X$,
  if $x_k \geq y_k$ for all $k$, then $x \succeq y$, and
  if $x_k > y_k$ for all $k$, then $x > y$.

The relation $\succeq$ satisfies *monotonicity* if it satisfies monotonicity at every $y \in X$.

**Proposition:** Any preference relation satisfying monotonicity and continuity can be represented by a utility function
Proof

- Take any bundle \( x \in \mathbb{R}^n_+ \).
- It is at least as good as the bundle \( 0 = (0, ..., 0) \).
- On the other hand \( M = (\max_k \{x_k\}, ..., \max_k \{x_k\}) \) is at least as good as \( x \).
- Both \( 0 \) and \( M \) are on the main diagonal.
- By continuity there is a bundle on the main diagonal that is indifferent to \( x \).
- By monotonicity this bundle is unique, denote it by \( (t(x), ..., t(x)) \).
- Let \( u(x) = t(x) \). We show that \( u \) represents the preferences:
  - By transitivity, \( x \succeq y \iff (t(x), ..., t(x)) \succeq (t(y), ..., t(y)) \).
  - By monotonicity this is true if and only if \( t(x) \geq t(y) \).

QED