Advanced Macroeconomics
The Solow Model

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Introduction

- Authors: Robert Solow (1956) and Trevor Swan (1956)
- Very simple framework for analyzing the mechanics of economic growth
- Engines of economic growth:
  - Technological change (exogenous)
  - Capital accumulation (endogenous)
- NOT a framework for explaining deep sources of economic growth
- Departure point for more elaborate analyses of growth
- Our discussion:
  - General equilibrium framework
  - Discrete time
Basic setup

- Closed economy
- No government
- One homogeneous final good
- Price of the final good normalized to 1 in each period (all variables expressed in real terms)
- Two types of agents in the economy:
  - Firms
  - Households
- Firms and households identical: one can focus on a representative firm and a representative household, aggregation straightforward
Firms I

- Final output produced by competitive firms
- Production function

\[ Y_t = F(K_t, A_t L_t) \]  (1)

where:

- \( K_t \) - capital stock
- \( L_t \) - labour input (population)
- \( A_t \) - technology

- Capital and labour inputs rented from households
- Technology grows at a constant rate \( g > 0 \):

\[ A_{t+1} = (1 + g)A_t \]

- Remarks:
  - \( F \) must be neoclassical
  - Technology must be of labour-augmenting type (Harrod neutral)
  - Technology is available for free
Firms II

- Neoclassical production function $F$
  - Constant returns to scale (wrt. capital and labour)
  - Positive and decreasing marginal products of capital and labour
  - Inada conditions
  - (Essentiality)
- E.g. for given $L_t$ and $A_t$:  

![Graph showing the relationship between $Y_t$ and $K_t$.]
Firms III

- Maximization problem of firms:

\[
\max_{L_t, K_t} \left\{ F(K_t, A_t L_t) - W_t L_t - R_{K,t} K_t \right\}
\]

- Firms maximize their profits, taking factor prices (wage rate \( W_t \) and capital rental rate \( R_{K,t} \)) as given (competitive factor markets)

- First order conditions:

\[
W_t = \frac{\partial F}{\partial L_t}
\]

\[
R_{K,t} = \frac{\partial F}{\partial K_t}
\]

- Remarks:
  - Firms’ problem is static
  - Firms make zero profits
Households

- Own production factors (capital and labour), so earn income on renting them to firms
- Labour supplied inelastically, grows at a constant rate $n > 0$:
  \[ L_{t+1} = (1 + n)L_t \]
- Capital is accumulated from investment $I_t$ and subject to depreciation (at a constant rate $\delta > 0$):
  \[ K_{t+1} = (1 - \delta)K_t + I_t \] (4)
- Total income of households can be split between consumption or savings (equal to investment):
  \[ W_tL_t + R_{K,t}K_t = C_t + S_t = C_t + I_t \]
- Save a constant fraction $s$ of income $Y_t$ (do NOT optimize):
  \[ I_t = sY_t \quad C_t = (1 - s)Y_t \] (5)
General equilibrium

- Market clearing conditions:
  - Output produced by firms must be equal to households’ total spending (on consumption and investment):
    \[ Y_t = C_t + I_t \]
  - Labour supplied by households must be equal to labour input demanded by firms
  - Capital supplied by households must be equal to capital input demanded by firms

Definition of equilibrium

A sequence of \( \{K_t, Y_t, C_t, I_t, W_t, R_{K,t}\}_{t=0}^{\infty} \) for a given sequence of \( \{L_t, A_t\}_{t=0}^{\infty} \) and an initial capital stock \( K_0 \), such that: \( K_t \) satisfies (4), \( Y_t \) is given by (1), \( C_t \) and \( I_t \) are given by (5), and \( W_t \) and \( R_{K,t} \) are given by (2) and (3), respectively.
Production function in intensive form

- Using CRS assumption for $F$:
  \[ y_t \equiv \frac{Y_t}{A_tL_t} = \frac{1}{A_tL_t} F(K_t, A_tL_t) = F\left(\frac{K_t}{A_tL_t}, 1\right) = f(k_t) \]

- Variables divided by $A_tL_t$ are "per unit of effective labour"
- Model is easier to analyze if working with normalized variables
- $f'(k_t) = \frac{\partial F}{\partial K_t}$ (marginal product of capital)
- Factor prices ($W_t$ from Euler’s theorem):
  - $R_{K,t} = f'(k_t)$
  - $W_t = A_t (f(k_t) - f'(k_t)k_t)$
- Capital share in output is equal to output elasticity wrt. capital:
  - $\alpha(k_t) = \frac{R_{K,t}K_t}{Y_t} = \frac{\partial F}{\partial K_t} K_t = \frac{f'(k_t)k_t}{f(k_t)}$
- Labour share: $1 - \alpha(k_t)$
Properties of $f(k_t)$:

- $f'(k_t) > 0$
- $f''(k_t) < 0$
- Inada conditions: $\lim_{k_t \to 0} f'(k_t) = \infty \quad \lim_{k_t \to \infty} f'(k_t) = 0$
Fundamental equation of the Solow model

- Capital law of motion using normalized variables

\[ k_{t+1} = \frac{1 - \delta}{(1 + g)(1 + n)} k_t + \frac{s}{(1 + g)(1 + n)} f(k_t) \]  \hspace{1cm} (6)

- Graphically:
If \( k_t = k^* \) then:

\[
\frac{\Delta K_{t+1}}{K_t} = \frac{K_{t+1}}{K_t} - 1 = \frac{A_{t+1} L_{t+1} f(k^*)}{A_t L_t f(k^*)} - 1 = (1 + n)(1 + g) - 1 \approx n + g
\]

\[
\frac{\Delta Y_{t+1}}{Y_t} = \frac{A_{t+1} L_{t+1} f(k^*)}{A_t L_t f(k^*)} - 1 = (1 + n)(1 + g) - 1 \approx n + g
\]

\[
\frac{\Delta C_{t+1}}{C_t} = \text{do it yourself}
\]

\[
\frac{\Delta I_{t+1}}{I_t} = \text{do it yourself}
\]

\[
\frac{\Delta W_{t+1}}{W_t} = \text{do it yourself}
\]

\[
R_{K,t} = \text{do it yourself}
\]
Steady state equilibrium (balanced growth) II

Per capita $K_t$, $C_t$ and $Y_t$:

\[ \Delta \left( \frac{Y_{t+1}}{L_{t+1}} \right) = \frac{Y_{t+1}}{Y_t} - 1 = \frac{Y_{t+1}}{L_{t+1}} - 1 = \frac{(1+n)(1+g)}{(1+n)} - 1 = g \]

Factor shares (elasticities):

\[ \alpha(k^*) = \frac{f'(k^*)k^*}{f(k^*)} = const \]
Fundamental equation (6) rewritten:

$$\frac{\Delta k_{t+1}}{k_t} = \frac{1}{(1+g)(1+n)} \left[ s \frac{f(k_t)}{k_t} - (g + n + \delta + ng) \right]$$

(7)

Capital per unit of effective labour is constant in steady state, which implies:

$$s \frac{f(k^*)}{k^*} = g + n + \delta + ng$$

(8)

We can use this relationship to assess long-run implications of the Solow model.
Permanent change in the savings rate

- Differentiation of the long-run relationship (8) yields:

\[
\frac{\partial k^*}{\partial s} = \frac{k^*}{s(1 - \alpha(k^*))}
\]  

(9)

- Output elasticity wrt. the savings rate:

\[
\frac{\partial f(k^*)}{\partial s} \frac{s}{f(k^*)} = \frac{s}{f(k^*)} f'(k^*) \frac{\partial k^*}{\partial s}
\]  

(10)

- Using (9) and (10), after rearrangement, yields:

\[
\frac{\partial f(k^*)}{\partial s} \frac{s}{f(k^*)} = \frac{\alpha(k^*)}{1 - \alpha(k^*)}
\]  

(11)

The long-run effect of a change in \( s \) depends positively on \( \alpha(k^*) \)
Golden rule of capital accumulation

- What is the savings rate maximizing the steady-state consumption?
  - Steady state consumption is equal to
    \[ c^* = (1 - s)f(k^*) \] (12)
  - FOC obtained by differentiating RHS of this equation wrt. \( s \):
    \[ (1 - s)f'(k^*) \frac{\partial k^*}{\partial s} - f(k^*) = 0 \] (13)
  - Using (9) and rearranging yields:
    \[ s = \alpha(k^*) \]

- **Steady state consumption is at the highest level if** \( s = \alpha(k^*) \)
Steady state equilibrium is a stable long-run equilibrium
Conditional convergence

$$\frac{\Delta k_{t+1}}{k_t} = \frac{1}{(1 + g)(1 + n)} \left[ s \frac{f(k_t)}{k_t} - (g + n + \delta + ng) \right]$$
First-order approximation of the fundamental equation (7) around \( \ln k_t = \ln k^* \) (log-linearization):

\[
\ln \left( \frac{k_{t+1}}{k_t} \right) = \frac{1}{(1 + g)(1 + n)} \left[ s \frac{f(k_t)}{k_t} - (g + n + \delta + ng) \right] = \\
\frac{s}{(1 + g)(1 + n)} \frac{f'(k^*) (k^*)^2 - f(k^*)k^*}{(k^*)^2} (\ln k_t - \ln k^*) = \\
\frac{s}{(1 + g)(1 + n)} \frac{f(k^*)}{k^*} \left( \frac{f'(k^*)k^*}{f(k^*)} - 1 \right) (\ln k_t - \ln k^*) = \\
g + n + \delta + ng \frac{1}{(1 + g)(1 + n)} (\alpha(k^*) - 1) \ln \left( \frac{k_t}{k^*} \right)
\]

(14)

**Speed of convergence depends negatively on** \( \alpha(k^*) \)
We can write equation (14) in compact form as:

\[
\ln \left( \frac{k_{t+1}}{k_t} \right) = -\omega \ln \left( \frac{k_t}{k^*} \right)
\]

(15)

where: \( \omega = \frac{g+n+\delta+ng}{(1+g)(1+n)} (1 - \alpha(k^*)) \)

Equation (15) applies also for \( y_t \):

\[
\ln \left( \frac{y_{t+1}}{y_t} \right) = -\omega \ln \left( \frac{y_t}{y^*} \right)
\]

(16)

Equations (15) (or (16)) imply that each period 100\( \omega \) per cent of the gap between capital (or output) and its steady-state level disappears:

\[
\ln \left( \frac{y_{t+1}}{y^*} \right) = (1 - \omega) \ln \left( \frac{y_t}{y^*} \right)
\]

(17)
Main implications of the Solow model

- Long-run growth (of output per capita) possible only with technological progress
  - But: technology exogenous in the model
- We should observe conditional, but not necessarily unconditional, convergence
  - In line with the data
- For standard values of parameters (taken from the data), in particular $\alpha(k^*) \approx 0.33$:
  - Elasticity of output wrt. savings rate rather low
    - Far too low to explain differences in income per capita in the world
    - Differences in technology important
  - Speed of convergence rather high
    - Implies too high pace of catching-up
- Better fit if capital is defined in broader sense, i.e. $K_t$ also includes human capital so that $\alpha(k^*) \approx 0.66$