HJM model driven by Lévy process

Rafał Marcin Łochowski

Seminar on mathematical finance and actuarial mathematics
Vienna University of Technology
January 2006

Some definitions

- $P(t, \theta)$, $0 \leq t \leq \theta$, - market price at moment t of a (zero-coupon) bond paying 1 at the maturity time θ ($P(\theta, \theta) = 1$)

Some definitions

- $P(t,\theta)$, $0 \leq t \leq \theta$, - market price at moment t of a (zero-coupon) bond paying 1 at the maturity time θ ($P(\theta,\theta) = 1$)
- $f(t,\theta)$, $\theta \geq t$, - the forward rate curve, a function such that

$$P(t,\theta) = e^{-\int_t^\theta f(t,s)ds}$$

Some definitions

- $P(t, \theta)$, $0 \leq t \leq \theta$, - market price at moment t of a (zero-coupon) bond paying 1 at the maturity time θ ($P(\theta, \theta) = 1$)
- $f(t, \theta)$, $\theta \geq t$, - the forward rate curve, a function such that

$$P(t, \theta) = e^{-\int_t^\theta f(t,s)ds}$$

- In the context of the bond market, price curve $P(\cdot, \theta)$ is an element of some functional (infinitely dimensional) space

Dynamics of the forward rates

- Heath, Jarrow and Morton proposed to model the forward curves as Itô processes

$$df(t,\theta) = \alpha(t,\theta)dt + <\sigma(t,\theta), dW(t)>, 0 \leq t \leq \theta$$

with W - d-dimensional Wiener process

Dynamics of the forward rates

- Heath, Jarrow and Morton proposed to model the forward curves as Itô processes

$$df(t,\theta) = \alpha(t,\theta)dt + < \sigma(t,\theta), dW(t) >, 0 \leq t \leq \theta$$

with W - d-dimensional Wiener process

- More flexible but yet analytically tractable is an HJM model driven by a (possibly time-inhomogeneous) Lévy process

$$df(t,\theta) = \alpha(t,\theta)dt + < \sigma(t,\theta), dZ(t) >, 0 \leq t \leq \theta$$

The driving process Z

- Models with Z - a Lévy process with values in $\mathbb{R}^d$ were studied for example by Eberlein, Jacod and Raible (2005)

The driving process Z

- Models with Z - a Lévy process with values in $\mathbb{R}^d$ were studied for example by Eberlein, Jacod and Raible (2005)
- In general, we will assume that Z is a Lévy process with values in a separable Hilbert space U

The driving process Z

- Models with Z - a Lévy process with values in $\mathbb{R}^d$ were studied for example by Eberlein, Jacod and Raible (2005)
- In general, we will assume that Z is a Lévy process with values in a separable Hilbert space U
- Models of similar types have been already studied in papers by Björk, DiMasi, Kabanov and Runggaldier (1997), Eberlein and Railble (1999) or Özkan and Schmidt (2005)

The driving process Z

- Models with Z - a Lévy process with values in $\mathbb{R}^d$ were studied for example by Eberlein, Jacod and Raible (2005)
- In general, we will assume that Z is a Lévy process with values in a separable Hilbert space U
- Models of similar types have been already studied in papers by Björk, DiMasi, Kabanov and Runggaldier (1997), Eberlein and Railble (1999) or Özkan and Schmidt (2005)
- This talk is based on the paper by Jakubowski and Zabczyk (2004)

The goal

- The goal ot the talk is to state necessary and sufficient conditions, in terms of characteristics of the Lévy process Z, implying that the discounted bond price processes $\hat{P}(\cdot, \theta), \theta \in [0, T]$ are local martingales (see Delbaen and Schachermayer(1994)) and to derive the HJM type condition

The goal

- The goal ot the talk is to state necessary and sufficient conditions, in terms of characteristics of the Lévy process Z, implying that the discounted bond price processes $\hat{P}(\cdot, \theta), \theta \in [0, T]$ are local martingales (see Delbaen and Schachermayer(1994)) and to derive the HJM type condition
- It will turn out that model implies existence of exponential moments of the noise processes

Forward rate function revisited

- Let $r(t)$, $t \geq 0$ be a short rate process, i. e. if at moment 0 one puts into the bank account 1 then at moment t one has

$$B_t = e^{\int_0^t r(s)ds}$$

Forward rate function revisited

- Let $r(t)$, $t \geq 0$ be a short rate process, i. e. if at moment 0 one puts into the bank account 1 then at moment t one has

$$B_t = e^{\int_0^t r(s)ds}$$

- It is convenient to assume that once a bond has matured its money equivalent goes directly to the bank account. Thus $P(t, \theta)$ - the market price at moment t of a bond paying 1 at the maturity time θ, is defined also for $t > \theta$ by the formula

$$P(t, \theta) = e^{\int_\theta^t r(s)ds} = e^{-\int_t^\theta r(s)ds}$$

Forward rate function revisited

- Let $r(t)$, $t \geq 0$ be a short rate process, i. e. if at moment 0 one puts into the bank account 1 then at moment t one has
$$B_t = e^{\int_0^t r(s)ds}$$
- It is convenient to assume that once a bond has matured its money equivalent goes directly to the bank account. Thus $P(t, \theta)$ - the market price at moment t of a bond paying 1 at the maturity time θ, is defined also for $t > \theta$ by the formula
$$P(t, \theta) = e^{\int_\theta^t r(s)ds} = e^{-\int_t^\theta r(s)ds}$$
- Consequently, if for each $\theta > 0$, the forward rate function $f(t, \theta), t > \theta$ is constant in t and equals $r(\theta)$ then for any $t, \theta \in [0, T]$
$$P(t, \theta) = e^{-\int_t^\theta f(t,s)ds}$$

Discounted bond price process and HJM condition

- The discounted bond price process $\hat{P}(\cdot, \theta), \theta \in [0, T]$ is defined by the formula

$$\begin{aligned}
\hat{P}(t, \theta) &= P(t, \theta)/B_t \\
&= e^{-\int_t^\theta f(t,s)ds} e^{-\int_0^t r(s)ds} \\
&= e^{-\int_t^\theta f(t,s)ds} e^{-\int_0^t f(t,s)ds} \\
&= \exp(-\int_0^\theta f(t, s)ds)
\end{aligned}$$

Discounted bond price process and HJM condition

- The discounted bond price process $\hat{P}(\cdot, \theta), \theta \in [0, T]$ is defined by the formula

$$\begin{aligned} \hat{P}(t,\theta) &= P(t,\theta)/B_t \\ &= e^{-\int_t^\theta f(t,s)ds} e^{-\int_0^t r(s)ds} \\ &= e^{-\int_t^\theta f(t,s)ds} e^{-\int_0^t f(t,s)ds} \\ &= \exp(-\int_0^\theta f(t,s)ds) \end{aligned}$$

- The HJM postulate is the requirement that all the discounted bond price processes $\hat{P}(\cdot, \theta), \theta \in [0, T]$ are local martingales

Dynamics of the forward rate function revisited

- Recall the model driven by a Lévy process

$$df(t,\theta) = \alpha(t,\theta)dt + < \sigma(t,\theta), dZ(t) >, 0 \le t \le \theta$$

For each $\theta > 0$ the processes $\alpha(t,\theta), \sigma(t,\theta)$ will be assumed to be adapted with respect to a given filtration $(\mathcal{F}_t)$ and such that integrals $\int_0^t \alpha(s,\theta)ds$, $\int_0^t < \sigma(s,\theta), dZ(s) >$ are well defined

Dynamics of the forward rate function revisited

- Recall the model driven by a Lévy process

$$df(t,\theta) = \alpha(t,\theta)dt + < \sigma(t,\theta), dZ(t) >, 0 \le t \le \theta$$

  For each $\theta > 0$ the processes $\alpha(t,\theta), \sigma(t,\theta)$ will be assumed to be adapted with respect to a given filtration $(\mathcal{F}_t)$ and such that integrals $\int_0^t \alpha(s,\theta)ds$, $\int_0^t < \sigma(s,\theta), dZ(s) >$ are well defined
- Since for $t > \theta$ the forward rate is constant in t for $t > \theta$ we put

$$\alpha(t,\theta) = \sigma(t,\theta) = 0$$

Short rate function revisited

- hence for $0 \leq t \leq \theta \leq T$ we have

$$f(t,\theta) = f(0,\theta) + \int_0^t \alpha(s,\theta)ds + \int_0^t < \sigma(s,\theta), dZ(s) >$$

and for $\theta < t \leq T$

$$f(t,\theta) \quad = \quad f(0,\theta) + \int_0^t \alpha(s,\theta)ds + \int_0^t < \sigma(s,\theta), dZ(s) >$$

Short rate function revisited

- hence for $0 \leq t \leq \theta \leq T$ we have

$$f(t,\theta) = f(0,\theta) + \int_0^t \alpha(s,\theta)ds + \int_0^t < \sigma(s,\theta), dZ(s) >$$

and for $\theta < t \leq T$

$$\begin{aligned} f(t,\theta) &= f(0,\theta) + \int_0^t \alpha(s,\theta)ds + \int_0^t < \sigma(s,\theta), dZ(s) > \\ &= f(0,\theta) + \int_0^\theta \alpha(s,\theta)ds + \int_0^\theta < \sigma(s,\theta), dZ(s) > \end{aligned}$$

Short rate function revisited

- hence for $0 \leq t \leq \theta \leq T$ we have

$$f(t,\theta) = f(0,\theta) + \int_0^t \alpha(s,\theta)ds + \int_0^t < \sigma(s,\theta), dZ(s) >$$

and for $\theta < t \leq T$

$$\begin{aligned} f(t,\theta) &= f(0,\theta) + \int_0^t \alpha(s,\theta)ds + \int_0^t < \sigma(s,\theta), dZ(s) > \\ &= f(0,\theta) + \int_0^\theta \alpha(s,\theta)ds + \int_0^\theta < \sigma(s,\theta), dZ(s) > \\ &= f(\theta,\theta) \end{aligned}$$

Short rate function revisited

- hence for $0 \leq t \leq \theta \leq T$ we have

$$f(t,\theta) = f(0,\theta) + \int_0^t \alpha(s,\theta)ds + \int_0^t < \sigma(s,\theta), dZ(s) >$$

and for $\theta < t \leq T$

$$\begin{aligned} f(t,\theta) &= f(0,\theta) + \int_0^t \alpha(s,\theta)ds + \int_0^t < \sigma(s,\theta), dZ(s) > \\ &= f(0,\theta) + \int_0^\theta \alpha(s,\theta)ds + \int_0^\theta < \sigma(s,\theta), dZ(s) > \\ &= f(\theta,\theta) \end{aligned}$$

- Since for $t > \theta$ the forward rate $f(t,\theta)$ is constant in t and equals $r(\theta)$

$$r(\theta) = f(\theta,\theta)$$

The forward rate - a functional approach

- Let us assume, that for all $t \in [0, T]$ the function $f(t) : [0, T] \to \mathbb{R}$,

$$f(t)(\theta) := f(t, \theta)$$

is square integrable, i. e. is an element of $L^2[0, T]$

The forward rate - a functional approach

- Let us assume, that for all $t \in [0, T]$ the function $f(t) : [0, T] \to \mathbb{R}$,

$$f(t)(\theta) := f(t, \theta)$$

  is square integrable, i. e. is an element of $L^2[0, T]$
- Let us define $\alpha(t) \in L^2[0, T]$

$$\alpha(t)(\theta) := \alpha(t, \theta)$$

The forward rate - a functional approach

- Let us assume, that for all $t \in [0, T]$ the function $f(t) : [0, T] \to \mathbb{R}$,

$$f(t)(\theta) := f(t, \theta)$$

  is square integrable, i. e. is an element of $L^2[0, T]$
- Let us define $\alpha(t) \in L^2[0, T]$

$$\alpha(t)(\theta) := \alpha(t, \theta)$$

  and a linear map $\tilde{\sigma}(t) : U \to L^2[0, T]$

$$(\tilde{\sigma}(t)u)(\theta) :=< \sigma(t, \theta), u >$$

The forward rate - a functional approach

- Let us assume, that for all $t \in [0, T]$ the function $f(t) : [0, T] \to \mathbb{R}$,

$$f(t)(\theta) := f(t, \theta)$$

  is square integrable, i. e. is an element of $L^2[0, T]$
- Let us define $\alpha(t) \in L^2[0, T]$

$$\alpha(t)(\theta) := \alpha(t, \theta)$$

  and a linear map $\tilde{\sigma}(t) : U \to L^2[0, T]$

$$(\tilde{\sigma}(t)u)(\theta) :=< \sigma(t, \theta), u >$$

- The dynamics of $f(t)$ may be stated in the form

$$df(t) = \alpha(t)dt + \tilde{\sigma}(t)dZ(t)$$

The discounted bond price process as a semimartingale

- Let $\langle \cdot, \cdot \rangle$ denote the scalar product in $L^2[0, T]$

The discounted bond price process as a semimartingale

- Let $\langle \cdot, \cdot \rangle$ denote the scalar product in $L^2[0, T]$
- Let us fix $\theta \in [0, T]$ and denote $g_\theta = \chi_{[0,\theta]}$ - the characteristic function of the interval $[0, \theta]$

The discounted bond price process as a semimartingale

- Let $\langle \cdot, \cdot \rangle$ denote the scalar product in $L^2[0, T]$
- Let us fix $\theta \in [0, T]$ and denote $g_\theta = \chi_{[0,\theta]}$ - the characteristic function of the interval $[0, \theta]$
- Since

$$\int_0^\theta f(t, s) ds = \langle f(t), g_\theta \rangle$$

The discounted bond price process as a semimartingale

- Let $\langle \cdot, \cdot \rangle$ denote the scalar product in $L^2[0, T]$
- Let us fix $\theta \in [0, T]$ and denote $g_\theta = \chi_{[0,\theta]}$ - the characteristic function of the interval $[0, \theta]$
- Since

$$\int_0^\theta f(t,s)ds = \langle f(t), g_\theta \rangle$$

- we have

$$\hat{P}(t,\theta) = \exp(-\langle f(t), g_\theta \rangle)$$

The discounted bond price process as a semimartingale

- Let $\langle \cdot, \cdot \rangle$ denote the scalar product in $L^2[0, T]$
- Let us fix $\theta \in [0, T]$ and denote $g_\theta = \chi_{[0,\theta]}$ - the characteristic function of the interval $[0, \theta]$
- Since
$$\int_0^\theta f(t,s)ds = \langle f(t), g_\theta \rangle$$
- we have
$$\hat{P}(t,\theta) = \exp(-\langle f(t), g_\theta \rangle)$$
so the process $\hat{P}(t,\theta), t \in [0, T]$ is a semimartingale and one can find its decomposition using Itô's formula

First step - decomposition of the driving Lévy process Z

- Z is a cadlag process with independent stationary increments having values in a separable Hilbert space U

First step - decomposition of the driving Lévy process Z

- Z is a cadlag process with independent stationary increments having values in a separable Hilbert space U
- Let μ denote the measure associated to jumps of Z, i. e. for $\Gamma \in \mathcal{B}(U), \bar{\Gamma} \in U \setminus \{0\}$

$$\mu([0,t],\Gamma) = \sum_{0<s\leq t} \chi_\Gamma(\Delta Z(s)) := \sum_{0<s\leq t} \chi_\Gamma(Z(s) - Z(s-))$$

First step - decomposition of the driving Lévy process Z

- Z is a cadlag process with independent stationary increments having values in a separable Hilbert space U
- Let μ denote the measure associated to jumps of Z, i. e. for $\Gamma \in \mathcal{B}(U), \bar{\Gamma} \in U \setminus \{0\}$

$$\mu([0,t],\Gamma) = \sum_{0<s\leq t} \chi_\Gamma(\Delta Z(s)) := \sum_{0<s\leq t} \chi_\Gamma(Z(s) - Z(s-))$$

- let ν denote the Lévy measure of Z, i. e.

$$\mathbb{E}\mu([0,t],\Gamma) = t\nu(\Gamma)$$

First step - decomposition of the driving Lévy process Z

- Z is a cadlag process with independent stationary increments having values in a separable Hilbert space U
- Let μ denote the measure associated to jumps of Z, i. e. for $\Gamma \in \mathcal{B}(U), \bar{\Gamma} \in U \setminus \{0\}$

$$\mu([0,t],\Gamma) = \sum_{0<s\leq t} \chi_\Gamma(\Delta Z(s)) := \sum_{0<s\leq t} \chi_\Gamma(Z(s) - Z(s-))$$

- let ν denote the Lévy measure of Z, i. e.

$$\mathbb{E}\mu([0,t],\Gamma) = t\nu(\Gamma)$$

- with

$$\int_U (|x|^2 \wedge 1)\nu(dx) < \infty$$

First step - decomposition of the driving Lévy process Z

- Z is a cadlag process with independent stationary increments having values in a separable Hilbert space U
- Let μ denote the measure associated to jumps of Z, i. e. for $\Gamma \in \mathcal{B}(U), \bar{\Gamma} \in U \setminus \{0\}$

$$\mu([0,t],\Gamma) = \sum_{0<s\leq t} \chi_\Gamma(\Delta Z(s)) := \sum_{0<s\leq t} \chi_\Gamma(Z(s) - Z(s-))$$

- let ν denote the Lévy measure of Z, i. e.

$$\mathbb{E}\mu([0,t],\Gamma) = t\nu(\Gamma)$$

- with

$$\int_U (|x|^2 \wedge 1)\nu(dx) < \infty$$

- $\mu(ds,dx) - sd\nu(dx)$ is a compensated jump measure

Decomposition of the driving Lévy process Z continued

- The characteristic function of $Z(t)$ has the following form $\mathbb{E}e^{i<\lambda,Z(t)>} = e^{t\psi(\lambda)}$, where

$$\begin{aligned}\psi(\lambda) &= i < a, \lambda > -\frac{1}{2} < Q\lambda, \lambda > \\ &\quad + \int_U (e^{i<\lambda,x>} - 1 - i < \lambda, x > \chi_{|x|\leq 1})\nu(dx)\end{aligned}$$

Decomposition of the driving Lévy process Z continued

- The characteristic function of $Z(t)$ has the following form $\mathbb{E}e^{i<\lambda,Z(t)>} = e^{t\psi(\lambda)}$, where

$$\psi(\lambda) = i < a,\lambda > -\frac{1}{2} < Q\lambda,\lambda > + \int_U (e^{i<\lambda,x>} - 1 - i < \lambda, x > \chi_{|x|\le 1})\nu(dx)$$

- moreover Z has a decomposition

$$Z(t) = at + W(t) + \int_0^t \int_{|y|\le 1} y(\mu(ds,dy) - ds\nu(dy)) + \int_0^t \int_{|y|>1} y\mu(ds,dy)$$

where W is a Wiener process having values in U with covariance operator Q

Second step - decomposition of the process $\langle g_\theta, f(t) \rangle$

- Following the dynamics of $f(t)$ and the decomposition of Z we get

Second step - decomposition of the process $\langle g_\theta, f(t)\rangle$

- Following the dynamics of $f(t)$ and the decomposition of Z we get

$$
\begin{aligned}
d\langle g_\theta, f(t)\rangle &= \langle g_\theta, df(t)\rangle \\
&= \langle g_\theta, \alpha(t)\rangle dt + \langle g_\theta, \tilde{\sigma}(t) dZ(t)\rangle \\
&= \langle g_\theta, \alpha(t)\rangle dt + < \tilde{\sigma}^*(t) g_\theta, dZ(t) > \\
&= \langle g_\theta, \alpha(t)\rangle dt + < \tilde{\sigma}^*(t) g_\theta, a dt + dW(t) > \\
&\quad + < \tilde{\sigma}^*(t) g_\theta, \int_U \chi_{\{|y|\le 1\}} y(\mu(dt, dy) - dt\nu(dy)) > \\
&\quad + < \tilde{\sigma}^*(t) g_\theta, \int_U \chi_{\{|y|>1\}} y\mu(dt, dy) >
\end{aligned}
$$

Itô formula

- To ease the notation let us denote $X(t) = \langle g_\theta, f(t) \rangle$, $\Sigma(t) = \tilde{\sigma}^*(t) g_\theta$ (recall that θ is fixed)

Itô formula

- To ease the notation let us denote $X(t) = \langle g_\theta, f(t)\rangle$, $\Sigma(t) = \tilde{\sigma}^*(t) g_\theta$ (recall that θ is fixed)
- Itô formula for the real semimartingale $X(t)$ and real C^2 function ξ has the following form

$$\begin{aligned}
\xi(X(t)) &= \xi(X(0)) + \int_0^t \xi'(X(s-))dX(s) \\
&+ \frac{1}{2}\int_0^t \xi''(X(s)) < Q\Sigma(s), \Sigma(s) > ds \\
&+ \sum_{s \le t} [\xi(X(s)) - \xi(X(s-)) - \xi'(X(s-))\Delta X(s)] \\
&= \xi(X(0)) + I_1(t) + I_2(t) + I_3(t)
\end{aligned}$$

Calculations continued

- We have

$$
\begin{aligned}
I_1(t) &= \int_0^t \xi'(X(s-))dX(s) \\
&= \int_0^t \xi'(X(s-))[\langle g_\theta, \alpha(s)\rangle + < \Sigma(s), a >]ds \\
&\quad + M_1(t) \\
&\quad + \int_0^t \int_U \chi_{\{|y|>1\}} \xi'(X(s-)) < \Sigma(s), y > \mu(ds, dy)
\end{aligned}
$$

  where $M_1(t)$ is a local martingale as a sum of a Wiener integral and integral with respect to the compensated jump measure $\mu(ds, dy) - ds\nu(dy)$

Calculations continued

- Let us further denote by μ_X the jump measure of the semimartingale X. Since $\Delta X(t) =< \Sigma(t), \Delta Z(t) >$

$$\begin{aligned}\mu_X([0,T],\Gamma) &= \sum_{s\le t} \chi_\Gamma(< \Sigma(t), \Delta Z(t) >) \\ &= \int_0^t \int_U \chi_\Gamma(< \Sigma(s), y >)\mu(ds, dy)\end{aligned}$$

Calculations continued

- Let us further denote by μ_X the jump measure of the semimartingale X. Since $\Delta X(t) =< \Sigma(t), \Delta Z(t) >$

$$\begin{aligned} \mu_X([0,T],\Gamma) &= \sum_{s\leq t} \chi_\Gamma(< \Sigma(t), \Delta Z(t) >) \\ &= \int_0^t \int_U \chi_\Gamma(< \Sigma(s), y >)\mu(ds, dy) \end{aligned}$$

- therefore

$$\begin{aligned} I_3(t) &= \sum_{s\leq t}[\xi(X(s)) - \xi(X(s-)) - \xi'(X(s-))\Delta X(s)] \\ &= \int_0^t \int_U [\xi(X(s-)+ < \Sigma(s), y >) - \xi(X(s-)) \\ &\quad -\xi'(X(s-)) < \Sigma(s), y >]\mu(ds, dy) \end{aligned}$$

Calculations continued

- Consequently

$$
\begin{aligned}
\xi(X(t)) &= \xi(X(0)) + M_2(t) \\
&+ \int_0^t \xi'(X(s-))[\langle g_\theta, \alpha(s)\rangle + < \Sigma(s), a >]ds \\
&+ \frac{1}{2}\int_0^t \xi''(X(s)) < Q\Sigma(s), \Sigma(s) > ds \\
&+ \int_0^t \int_U [\xi(X(s-) + < \Sigma(s), y >) - \xi(X(s-)) \\
&\quad - \chi_{\{|y|\leq 1\}}\xi'(X(s-)) < \Sigma(s), y >] ds\nu(dy)
\end{aligned}
$$

where M_2 is a new local martingale: a sum of M_1 and again an integral with respect to the compensated measure $\mu(ds, dy) - ds\nu(dy)$

Calculations finished

- In order $\xi(X(t))$ to be a local martingale we sholud have for all $t \in [0, T]$

$$
\begin{aligned}
\xi'(X(t-))[\langle g_\theta, \alpha(t)\rangle + < \Sigma(t), a >] \\
+ \frac{1}{2}\xi''(X(t)) < Q\Sigma(t), \Sigma(t) > \\
+ \int_U [\xi(X(t-) + < \Sigma(t), y >) - \xi(X(t-)) \\
- \chi_{\{|y|\leq 1\}}\xi'(X(t-)) < \Sigma(t), y >]\nu(dy) \equiv 0
\end{aligned}
$$

Calculations finished

- In order $\xi(X(t))$ to be a local martingale we sholud have for all $t \in [0, T]$

$$
\begin{aligned}
&\xi'(X(t-))[\langle g_\theta, \alpha(t)\rangle + < \Sigma(t), a >] \\
&\qquad + \frac{1}{2}\xi''(X(t)) < Q\Sigma(t), \Sigma(t) > \\
&\qquad + \int_U [\xi(X(t-) + < \Sigma(t), y >) - \xi(X(t-)) \\
&\qquad - \chi_{\{|y| \le 1\}}\xi'(X(t-)) < \Sigma(t), y >]\nu(dy) \equiv 0
\end{aligned}
$$

Recall that $\hat{P}(t,\theta) = \xi(X(t))$ with $\xi(x) = e^{-x}$, so finally we get

$$
\begin{aligned}
&-\langle g_\theta, \alpha(t)\rangle - < \Sigma(t), a > + \frac{1}{2} < Q\Sigma(t), \Sigma(t) > \\
&\quad + \int_U [e^{-<\Sigma(t),y>} - 1 + \chi_{\{|y| \le 1\}} < \Sigma(t), y >]\nu(dy) \equiv 0
\end{aligned}
$$

More elegant formula

- If we define for all $u \in U$

$$J(u) = -< u, a > + \frac{1}{2} < Qu, u > + \int_U [e^{-<u,y>} - 1 + \chi_{\{|y| \leq 1\}} < u, y >] \nu(dy)$$

More elegant formula

- If we define for all $u \in U$

$$J(u) = - < u, a > + \frac{1}{2} < Qu, u > + \int_U [e^{-<u,y>} - 1 + \chi_{\{|y| \leq 1\}} < u, y >] \nu(dy)$$

(note that J may attain value $+\infty$)

More elegant formula

- If we define for all $u \in U$

$$J(u) = -< u, a > + \frac{1}{2} < Qu, u > + \int_U [e^{-<u,y>} - 1 + \chi_{\{|y| \le 1\}} < u, y >] \nu(dy)$$

(note that J may attain value $+\infty$) then the obtained condition reads

$$\int_t^\theta \alpha(t, v) dv = J\left(\int_t^\theta \sigma(t, v) dv\right), t \le \theta$$

Conclusion

Moreover, we have

Conclusion

Moreover, we have

Theorem

If HJM postulate is satisfied and for some closed set B of U and some $\theta, s > 0$, $supp(\tilde{\sigma}^*(s)g_\theta) \supseteq B$ *then for every c in some dense subset of B*

$$\int_{\{|y|>1\}} e^{<c,y>} dy < +\infty \tag{1}$$

Conclusion

Moreover, we have

Theorem

If HJM postulate is satisfied and for some closed set B of U and some $\theta, s > 0$, supp$(\tilde{\sigma}^*(s)g_\theta) \supseteq B$ then for every c in some dense subset of B

$$\int_{\{|y|>1\}} e^{<c,y>} dy < +\infty \tag{1}$$

Opposite, if (1) holds for all $u \in U$ then HJM postulate is satisfied iff

$$\int_t^\theta \alpha(t,v)dv = J(\int_t^\theta \sigma(t,v)dv), t \leq \theta$$

References

- Th. Björk, G. DiMasi, Y. Kabanov and W. Runggaldier, *Towards a general theorem of bond markets*, *Finance and Stochastics 1 (1997), 141 - 174*
- F. Delbaen, W. Schachermayer, *A general version of the fundamental theorem of asset pricing*, *Math. Annalen 300 (1994), 463 - 520*
- E. Eberlain, S. Raible, *Term structure models driven by general Lévy process*, *Math. Finance 9 (1999), 31 - 53*
- E. Eberlain, J. Jacod and S. Raible, *Lévy term structure models: no arbitrage and completness*, *Finance and Stochastics 9 (2005), 67 - 88*
- F. Özkan, Th. Schmidt, *Credit risk with infinite dimensional Lévy processes*, forthcoming in *Statistics and Decisions*
- J. Jakubowski, J. Zabczyk, *HJM condition for models with Lévy noise*, to appear