STOPPING TIME FOR ORNSTEIN-UHLENBECK PROCESS VS. STOPPING TIME FOR DISCRETE AR(1) PROCESS

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In [4] there was considered trading strategy based on hypothesis of existence of cointegrating relationship between series of prices of contracts for two commodities. More precisely, it was assumed that the cointegrating series has AR(1) structure and from this assumption long-run gain for (approximately) optimal trading strategy consisting of simultaneous selling and buying contracts for these commodities was determined (for some range of parameters characterizing AR(1) series). The formula for the approximate optimal gain was derived from reasoning based on heuristic assumption that the stopping time for AR(1) process may be approximated by stopping time for its continuous counterpart - Ornstein-Uhlenbeck process. This paper deals with more exact results concerning this approximation.

1. Introduction. Let $P, Q, ..., R$ be some commodities and $P_t, Q_t, ..., R_t$ denote the prices of the contracts for these commodities at time t.

Henceforth we will assume that there exists cointegrating relationship between processes $(P_t), (Q_t), ...$ and (R_t), i.e. there exist such constants $A, B, ..., C$ that the process

$$\begin{aligned} X_t &= [A, B, ..., C] \circ [P_t, Q_t, ..., R_t] \\ &= A \cdot P_t + B \cdot Q_t + ... + C \cdot R_t \end{aligned}$$

is a stationary one (in a strong sense).

Let a be such a positive constant that

$$P\left(\forall T \geq 0 \quad \exists t_a(T), t_{-a}(T) \geq T \mid X_{t_a(T)} \geq a, X_{t_{-a}(T)} \leq -a\right) = 1.$$

We consider the following trading strategy based on the assumption of stationarity of the process (X_t).

If X_t exceeds threshold a then we take short positions in the contracts for all commodities, prices of which correspond to the positive coefficients $A, B, ..., C$, and simultaneously take long positions in the contracts for all commodities, prices of which correspond to negative coefficients $A, B, ..., C$.

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If X_t decreases below $-a$, then we do opposite, after closing old positions. The amounts of contracts for commodities $P, Q, ..., R$, which we buy shall be equal (or proportional to) $|A|, |B|, ..., |C|$ respectively.

The pair of this tradings gives the profit which is equal (or proportional to) $2a$. But the nominal rate of return of this strategy depends on the time elapsed between two tradings: $T_{a,-a}(0) = T_{-a}(T_a(0)) - T_a(0)$, where

$$\begin{aligned} T_a(t) &= \inf\{T \geq t | X_T \geq a\}, \\ T_{-a}(t) &= \inf\{T \geq t | X_T \leq -a\}. \end{aligned}$$

We will consider the following models of the market:

1) continuous model, where parameter t attains all positive real values;

2) discrete model, where parameter t attains only values from the set $\delta\mathbb{Z}_+ = \{0, \delta, 2\delta, 3\delta, ...\}$ for some $\delta > 0$.

Moreover, we will assume exact structure of the process (X_t) for both models. Namely, we will assume that (X_t) is AR(1) process, when discrete model is considered and (X_t) is Ornstein-Uhlenbeck process, (which is continuous cunterpart for AR(1) process in some sense), when we consider continuous model.

In the next section we recall known or easily derivable quantitative facts about the distribution of $T_a(0)$ and $T_{a,-a}(0)$ for continous model.

The last section deals with the discrete time model and we prove there that the distribution of $T_a(0)$ and $T_{a,-a}(0)$ for discrete model converges to its continuous counterpart when $\delta \to 0_+$.

2. Continuous model. Let us assume that $(X_t)_{t\geq 0}$ is an Orstein-Uhlenbeck process. X_t may be introduced as unique strong solution of the following stochastic differential equation

$$dX_t = -\beta X_t dt + \alpha\sqrt{2\beta} dB_t, \tag{1}$$

where $\alpha, \beta > 0$ and $(B_t)_{t\geq 0}$ is a standard Wiener process (Brownian motion).

$(X_t)_{t\geq 0}$ has continuous trajectories.

In order to ease the notation we will assume that $\alpha\sqrt{2\beta} = 1$ (which is equivalent with some space scaling of the process X_t).

For $a \in R$ let us define

$$T_a = \inf\{t \geq 0 : X_t = a\}. \tag{2}$$

It is known (see [2]) that for $t > 0$

$$E\left(e^{-tT_a} | X_0 = x\right) = \frac{e^{\beta x^2/2} D_{-t/\beta}\left(\varepsilon x\sqrt{2\beta}\right)}{e^{\beta a^2/2} D_{-t/\beta}\left(\varepsilon a\sqrt{2\beta}\right)},$$

where $\varepsilon = sign\,(x-a)$ and D_ν stands for the parabolic cylinder function (see [1]) which admits for $\nu < 0$ the following integral representation

$$D_\nu(z) = \frac{e^{-z^2/4}}{\Gamma(-\nu)} \int_0^\infty u^{-\nu-1} \exp\left(-\frac{u^2}{2} - zu\right) du.$$

Now, since $(X_t)_{t\geq 0}$ has continuous trajectories we have

$$T_a(0) = \begin{cases} T_a \text{ if } a \geq X_0, \\ 0 \text{ if } a < X_0 \end{cases}$$

and if $X_0 \sim N(0, \alpha^2) = N\left(0, \frac{1}{2\beta}\right)$ is independent from $(B_t)_{t\geq 0}$ then

$$E\left(e^{-tT_a(0)}\right) = 1 - \Phi\left(\frac{a}{\alpha}\right) + \sqrt{\frac{\beta}{\pi}} \int_{-\infty}^a \frac{e^{\beta x^2/2} D_{-t/\beta}\left(\varepsilon x\sqrt{2\beta}\right)}{e^{\beta a^2/2} D_{-t/\beta}\left(\varepsilon a\sqrt{2\beta}\right)} e^{-\beta x^2} dx,$$

where Φ is a distribuant of standard normal distribution.

Remark. *If we assume that $X_0 \sim N(0,\alpha^2)$ is independent from $(B_t)_{t\geq 0}$ then $(X_t)_{t\geq 0}$ defined by (1) admits the following Doob's representation*

$$X_t = \alpha e^{-\beta t} W\left(e^{2\beta t}\right), \tag{3}$$

where $W(.)$ is a standard Wiener process.

From Laplace transform we may obtain $ET_a(0)$ and $VarT_a(0)$ but it is rather laborious task. More interesting are formulae for $E(T_{a,-a}(0))$ and $Var(T_{a,-a}(0))$. We have (compare [2] and [5])

$$\begin{aligned} E(T_{a,-a}(0)) &= \frac{\sqrt{\pi}}{\beta} \int_0^{a/\sqrt{2\beta}} e^{t^2/2} dt \\ &= \frac{\sqrt{\pi}}{\beta} \sum_{k=0}^\infty \frac{1}{4^k \cdot (k+1) \cdot k!} \left(\frac{a^2}{\beta}\right)^k \end{aligned}$$

and a bit more complicated formula for variance

$$\begin{aligned} Var(T_{a,-a}(0)) &= 8\sqrt{\frac{\pi}{\beta}} \int_{-a}^a \left(\int_{-\infty}^t \left(\int_u^a e^{-\beta(u^2-v^2-t^2)} dt\right) du\right) dv \\ &\quad - E(T_{a,-a}(0))^2. \end{aligned}$$

3. Discrete model. Henceforth, in order to distinguish continuous and discrete models, instead of writing X_t in the discrete case we will denote cointegrating series by $\tilde{X}_t$.

In the second model we will assume that $\left(\tilde{X}_t\right)$ has discrete AR(1) structure

$$\tilde{X}_{t+\delta} = \gamma \tilde{X}_t + \tilde{\varepsilon}_t,$$

where

- $\tilde{X}_0 \sim N\left(0, \alpha^2\right)$
- $\tilde{\varepsilon}_t, t \in \delta\mathbb{Z}_+$, are i.i.d. normal variables,
- for every $t \in \delta\mathbb{Z}_+$ variables $\tilde{\varepsilon}_t$ and $\tilde{X}_t$ are independent,
- $\gamma > 0$.

From this assumptions we have that $\left(\tilde{X}_t\right)_{t\in\delta\mathbb{Z}_+}$ is a stationary Markov chain if and only if $\gamma < 1$ and $\tilde{\varepsilon}_t \sim N\left(0, \alpha^2\left(1-\gamma^2\right)\right)$.

Let $\gamma = e^{-\beta\delta}$ for some $\beta > 0$, then $\tilde{\varepsilon}_t \sim N\left(0, \alpha^2\left(1-e^{-2\beta\delta}\right)\right)$ and it is easy to check that the same relationships hold for process defined by (3):

$$\begin{aligned} X_{t+\delta} &= \alpha e^{-\beta(t+\delta)} W\left(e^{2\beta(t+\delta)}\right) \\ &= \alpha e^{-\beta(t+\delta)} W\left(e^{2\beta t}\right) + \alpha e^{-\beta(t+\delta)}\left[W\left(e^{2\beta(t+\delta)}\right) - W\left(e^{2\beta t}\right)\right] \\ &= e^{-\beta\delta} X_t + \varepsilon_t, \end{aligned}$$

where $\varepsilon_t \sim N\left(0, \alpha^2\left(1-e^{-2\beta\delta}\right)\right)$ since

$$\begin{aligned} E\varepsilon_t^2 &= \alpha^2 e^{-2\beta(t+\delta)}\left[e^{2\beta(t+\delta)} - e^{2\beta t}\right] \\ &= \alpha^2 e^{-2\beta(t+\delta)} e^{2\beta(t+\delta)}\left[1 - e^{-2\beta\delta}\right] \\ &= \alpha^2\left(1-e^{-2\beta\delta}\right). \end{aligned}$$

Hence $\left(\tilde{X}_t\right)_{t\in\delta\mathbb{Z}_+}$ may be treated as the process (X_t) sampled at the points $t \in \delta\mathbb{Z}_+$.

Let us now denote

$$T_a^{(\delta)} = \inf\left\{t \in \delta\mathbb{Z}_+ : \tilde{X}_t \geq a\right\}.$$

Of course $T_a \leq T_a^{(\delta)}$, where T_a is defined by (2). We will show

THEOREM 1. *$T_a^{(\delta)} \to T_a$ in distribution, when $\delta \to 0_+$.*

In order to prove the above theorem, first we will prove the following lemmas.

LEMMA 1. *If $x, y \in R$, $c > 0$ then from inequalities $|x - y| \geq 2|y|$ and $x \leq c$ it follows that $x - y \leq 2c$.*

PROOF. We have three following possibilities

- $x < y$ then $x - y < 0 \leq 2c$.
- $y > 0$ then $x - y < x \leq c < 2c$.
- $x \geq y$ and $y \leq 0$ then $x - y = |x - y| \geq 2|y| = -2y$, hence $x \geq -y$ and $x - y \leq 2x \leq 2c$.

□

LEMMA 2. *Let $W(.)$ be a standard Wiener process, $\alpha > 0$ and $0 \leq t_1 \leq t_2 \leq ... \leq t_n$ be such a sequence that $t_{k+1} \geq (n^6 + 1) t_k$ and $t_{k+1} \geq t_k + \alpha$ for $k = 1, 2, ..., n - 1$. Then for every $c > 0$*

$$P(\max\{W(t_1), W(t_2), ..., W(t_n)\} \leq c) \leq \varepsilon_n(\frac{c}{\sqrt{\alpha}}),$$

where

$$\varepsilon_n(\frac{c}{\sqrt{\alpha}}) = \frac{2}{n} + 2ne^{-\frac{n^2}{8}} + \Phi\left(\frac{2c}{\sqrt{\alpha}}\right)^{n-1}.$$

PROOF. For $k = 1, 2, ..., n - 1$ let $c_k \in \left[n\sqrt{t_k}, \frac{\sqrt{t_{k+1} - t_k}}{n^2}\right]$ (since $t_{k+1} \geq (n^6 + 1) t_k$ we have $\frac{\sqrt{t_{k+1} - t_k}}{n^2} \geq \frac{\sqrt{n^6 t_k}}{n^2} = n\sqrt{t_k}$) and let G be a random variable with standard normal distribution. We have

$$\begin{aligned}
&P(|W(t_{k+1}) - W(t_k)| < 2|W(t_k)|) \\
&\leq \; P(|W(t_{k+1}) - W(t_k)| \leq c_k) + P(2|W(t_k)| > c_k) \\
&= \; P\left(|G| < \frac{c_k}{\sqrt{t_{k+1} - t_k}}\right) + P\left(|G| > \frac{c_k}{2\sqrt{t_k}}\right) \\
&\leq \; \frac{2c_k}{\sqrt{t_{k+1} - t_k}} + 2e^{-\frac{c_k^2}{8t_k}} \leq \frac{2}{n^2} + 2e^{-\frac{n^2}{8}}.
\end{aligned}$$

We used the estimates $P(|G| < u) \leq \frac{1}{\sqrt{2\pi}} 2u \leq 2u$ and $P(|G| > u) = 2P(G > u) \leq 2e^{-u^2} Ee^{uG} \leq 2e^{-u^2} e^{u^2/2} = 2e^{-u^2/2}$ which hold for $u > 0$.

From the previous lemma

$$
\begin{aligned}
&P\left(\max\left\{W\left(t_1\right), W\left(t_2\right), ..., W\left(t_n\right)\right\} \le c\right) \\
&\quad\le\ (n-1)\left(\frac{2}{n^2} + 2e^{-\frac{n^2}{8}}\right) \\
&\quad+\ P\left(W\left(t_{k+1}\right) \le c \ \&\ \left|W\left(t_{k+1}\right) - W\left(t_k\right)\right| \ge 2\left|W\left(t_k\right)\right| \text{ for } k = 1, 2, ..., n-1\right) \\
&\quad\le\ \frac{2}{n} + 2ne^{-\frac{n^2}{8}} + P\left(W\left(t_{k+1}\right) - W\left(t_k\right) \le 2c \text{ for } k = 1, 2, ..., n-1\right) \\
&\quad=\ \frac{2}{n} + 2ne^{-\frac{n^2}{8}} + \prod_{k=1}^{n-1} P\left(G \le \frac{2c}{\sqrt{t_{k+1} - t_k}}\right) \\
&\quad\le\ \frac{2}{n} + 2ne^{-\frac{n^2}{8}} + \Phi\left(\frac{2c}{\sqrt{\alpha}}\right)^{n-1}.
\end{aligned}
$$

□

Now we will estimate $P\left(T_a^\delta \ge T_a + \sqrt{\delta}\right)$. In order to ease notation we will assume that $X_t = e^{-t/2} W\left(e^t\right)$, since it is equivalent with space and time scaling ot the process (X_t). We have

$$
\begin{aligned}
&P\left(T_a^\delta \ge t + \sqrt{\delta} | T_a = t\right) \\
&\quad=\ P\left(\forall s \in \delta\mathbb{Z}_+ \cap \left[t, t + \sqrt{\delta}\right] \ X_s < a | T_a = t\right) \\
&\quad=\ P\left(\forall s \in \delta\mathbb{Z}_+ \cap \left[t, t + \sqrt{\delta}\right] \ X_s < a | X_t = a\right) \\
&\quad=\ P\left(\forall s \in \left(\delta\mathbb{Z}_+ - t\right) \cap \left[0, \sqrt{\delta}\right] \ X_s < a | X_0 = a\right) \\
&\quad=\ P\left(\forall s \in \left(\delta\mathbb{Z}_+ - t\right) \cap \left[0, \sqrt{\delta}\right] \ e^{-s/2} W\left(e^s\right) < a | W(1) = a\right) \\
&\quad=\ P\left(\forall u \in e^{\delta\mathbb{Z}_+ - t} \cap \left[1, e^{\sqrt{\delta}}\right] \ W(u) < a\sqrt{u} | W(1) = a\right) \\
&\quad\le\ P\left(\forall u \in e^{\delta\mathbb{Z}_+ - t} \cap \left[1, e^{\sqrt{\delta}}\right] \ W(u) < ae^{\sqrt{\delta}/2} | W(1) = a\right) \\
&\quad=\ P\left(\forall u \in e^{\delta\mathbb{Z}_+ - t} \cap \left[1, e^{\sqrt{\delta}}\right] \ W(u-1) < a\left(e^{\sqrt{\delta}/2} - 1\right)\right) \\
&\quad=\ P\left(\forall u \in \left(e^{\delta\mathbb{Z}_+ - t} - 1\right) \cap \left[0, e^{\sqrt{\delta}} - 1\right] \ W(u) < a\left(e^{\sqrt{\delta}/2} - 1\right)\right).
\end{aligned}
$$

Let us now take the smallest integer z_0 such that $\delta z_0 - t \ge \delta$. Then we have $\delta\left(z_0 - 1\right) - t < \delta$ and $\delta z_0 - t < 2\delta$. Defining $\alpha = e^{\delta z_0 - t}\delta$ and $s_k = e^{\delta z_0 - t + \delta k} - 1$ for $k = 1, 2, ..., N = \left\lfloor \frac{1}{2\sqrt{\delta}} \right\rfloor$ we get

$$
s_n = e^{\delta z_0 - t + \delta n} - 1 \le e^{2\delta} e^{\frac{1}{2}\delta\frac{1}{\sqrt{\delta}}} - 1 \le e^{\frac{1}{2}\sqrt{\delta} + 2\delta} - 1 \le e^{\sqrt{\delta}} - 1 \text{ for } \delta \le \frac{1}{16},
$$

$$\begin{aligned} s_{k+1}-s_k &= e^{\delta z_0-t+\delta(k+1)}-e^{\delta z_0-t+\delta k} \\ &= e^{\delta z_0-t}e^{\delta k}\left(e^{\delta}-1\right) \geq e^{\delta z_0-t}\delta=\alpha. \end{aligned}$$

Since $0\leq \delta z_0-t+\delta k\leq \sqrt{\delta}\leq \ln 2$ for $\delta\leq(\ln 2)^2$ and for $0\leq x\leq \ln 2$ we have $x\leq e^x-1\leq \frac{e^{\ln 2}-1}{\ln 2}x=\frac{1}{\ln 2}x$, hence

$$(4)\qquad (k+1)\,\delta\leq \delta z_0-t+\delta k\leq s_k=e^{\delta z_0-t+\delta k}-1\leq \frac{1}{\ln 2}\,(k+2)\,\delta.$$

Let now $n=n\,(N)$ be the greatest integer such that $\left(\left\lceil\frac{2}{\ln 2}\right\rceil\left(n^6+1\right)\right)^n\leq N$ and define $k_i=\left(\left\lceil\frac{2}{\ln 2}\right\rceil\left(n^6+1\right)\right)^i$ for $i=1,2,...,n$. Defining $t_i=s_{k_i}$ for $i=1,2,...,n$ we get

$$\frac{t_{i+1}}{t_i}\geq\frac{(k_{i+1}+1)\,\delta}{\frac{1}{\ln 2}\,(k_i+2)\,\delta}\geq\frac{k_{i+1}}{\frac{2}{\ln 2}k_i}=\frac{\left\lceil\frac{2}{\ln 2}\right\rceil}{\frac{2}{\ln 2}}\left(n^6+1\right)\geq n^6+1.$$

Hence for the sequence $t_1,...,t_n$ the assumptions of the previous lemma hold and we have

$$\begin{aligned} &P\left(T_a^\delta\geq t+\sqrt{\delta}|T_a=t\right) \\ &\leq\ P\left(\forall u\in\left(e^{\delta\mathbb{Z}_+-t}-1\right)\cap\left[0,e^{\sqrt{\delta}}-1\right]\ W\,(u)<a\left(e^{\sqrt{\delta}/2}-1\right)\right) \\ &\leq\ P\left(\max\left\{W\,(t_1)\,,W\,(t_2)\,,...,W\,(t_n)\right\}<a\left(e^{\sqrt{\delta}/2}-1\right)\right) \\ &\leq\ \varepsilon_n\left(\frac{a\left(e^{\sqrt{\delta}/2}-1\right)}{\sqrt{e^{\delta z_0-t}\delta}}\right) \end{aligned}$$

and since $\frac{2a\left(e^{\sqrt{\delta}/2}-1\right)}{\sqrt{e^{\delta z_0-t}\delta}}\leq\frac{2a\frac{1}{\ln 2}\frac{\sqrt{\delta}}{2}}{e^{\delta/2}\sqrt{\delta}}\leq\frac{2}{\ln 2}a$ we get

$$\begin{aligned} P\left(T_a^\delta\geq T_a+\sqrt{\delta}\right) &= \int_0^{+\infty}P\left(T_a^\delta\geq t+\sqrt{\delta}|T_a=t\right)P\left(T_a\in[t,t+dt)\right) \\ &\leq \int_0^{+\infty}\varepsilon_n\left(\frac{2}{\ln 2}a\right)P\left(T_a\in[t,t+dt)\right) \\ &= \varepsilon_n\left(\frac{2}{\ln 2}a\right). \end{aligned}$$

Since for $\delta\to 0_+$ we have $N=\left\lfloor\frac{1}{2\sqrt{\delta}}\right\rfloor\to\infty$, $n=n\,(N)\to\infty$ and $\varepsilon_n\left(\frac{2}{\ln 2}a\right)\to 0$, we get

$$\lim_{\delta\to 0_+}P\left(T_a\leq T_a^\delta\leq T_a+\sqrt{\delta}\right)=1$$

which is even stronger result than convergence in distribution.

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