Truncated variation of a stochastic process - its optimality for processes with càdlàg trajectories and its limit distributions for diffusions

Rafał Łochowski

Warsaw School of Economics

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Agenda

1. Definition and properties of truncated variation
2. "Strong law of large numbers" for truncated variation of a Brownian motion and continuous semimartingales
3. "Central limit theorem" for truncated variation of a Brownian motion
4. "Central limit theorem" for truncated variation of diffusions driven by a Brownian motion
5. Sketches of the proofs

Truncated variation - how it appears

Let $f : [a; b] \to \mathbb{R}$ be a càdlàg function and let $c > 0$
Question: what is the smallest total variation possible of a (càdlàg) function from the ball $\left\{ g : \|f - g\|_\infty \leq \frac{1}{2}c \right\}$?

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$$\inf_{g:\|f-g\|_\infty \le \frac{1}{2}c} TV(g, [a; b]) \ge TV^c(f, [a; b]),$$

where

$$TV(g, [a; b]) := \sup_n \sup_{a \le t_0 < t_1 < \ldots < t_n \le b} \sum_{i=1}^{n} |g(t_i) - g(t_{i-1})|,$$

$$TV^c(f, [a; b]) := \sup_n \sup_{a \le t_0 < t_1 < \ldots < t_n \le b} \sum_{i=1}^{n} \max\{|f(t_i) - f(t_{i-1})| - c, 0\},$$

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and follows immediately from the inequality

$$\left|g\left(t_i\right) - g\left(t_{i-1}\right)\right| \geq \max\left\{\left|f\left(t_i\right) - f\left(t_{i-1}\right)\right| - c, 0\right\}.$$

Truncated variation - definition and another interpretation

We will call the quantity

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truncated variation of the function f at the level c.

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truncated variation of the function f at the level c.
Trunated variation may be interpreted not only as the lower bound obtained on the previous slide but also as the variation taking into account only jumps greater than c.

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truncated variation of the function f at the level c.
Trunated variation may be interpreted not only as the lower bound obtained on the previous slide but also as the variation taking into account only jumps greater than c.
We will also show that in fact we have the equality

$$\inf_{g:\|f-g\|_\infty \le \frac{1}{2}c} TV(g,[a;b]) = TV^c(f,[a;b]).$$

Moreover, we will show that this lower bound is attainable, i.e. for some function f^c with $\|f - f^c\|_\infty \leq \frac{1}{2}c$ one has

$$\inf_{g:\|f-g\|_\infty \leq \frac{1}{2}c} TV\left(g, [a;b]\right) = TV\left(f^c, [a;b]\right) = TV^c\left(f, [a;b]\right).$$

Truncated variation - optimality

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$$\inf_{g:\|f-g\|_\infty \leq \frac{1}{2}c} TV(g,[a;b]) = TV(f^c,[a;b]) = TV^c(f,[a;b]).$$

Remark

Since every càdlàg function may be uniformly approximated with step functions, total variation of which is finite, hence TV^c is finite for every càdlàg function.

Optimal càdlàg function from the ball $\left\{g : \|f - g\|_\infty \leq \frac{1}{2}c\right\}$

We will show also that for any $c \leq \sup_{s,u\in[a;b]} |f(s) - f(u)|$ there exist an unique càdlàg function $f^c : [a; b] \to \mathbb{R}$ such that $\|f^c - f\|_\infty \leq \frac{1}{2}c$ and for any $s \in (a; b]$

$$TV\left(f^c, [a; s]\right) = TV^c\left(f, [a; s]\right).$$

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$$TV(f^c, [a;s]) = TV^c(f, [a;s]).$$

The function f^c appears to be a function starting from appropriate chosen starting point $f^c(a)$ and the most "lazy" function possible, which changes its value only if it is necessary for the relation $\|f - f^c\|_\infty \leq \frac{1}{2}c$ to hold.

The function f^c and two other functionals related - UTV and DTV

Moreover, we will show also, that function f^c may expressed with two other functionals related - upward and downward truncated variations.

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$$UTV^c(f,[a;b]) := \sup_n \sup_{a\le t_0<t_1<\ldots<t_n\le b} \sum_{i=1}^{n} \max\{f(t_i)-f(t_{i-1})-c,0\},$$

and downward truncated variation is defined with the formula

$$DTV^c(f,[a;b]) := \sup_n \sup_{a\le t_0<t_1<\ldots<t_n\le b} \sum_{i=1}^{n} \max\{f(t_{i-1})-f(t_i)-c,0\}.$$

TV and two others functionals related - UTV and DTV

Between TV, UTV and DTV the following relations hold

$$TV^c(f, [a;b]) = UTV^c(f, [a;b]) + DTV^c(f, [a;b]) \tag{1}$$

and

$$DTV^c(f, [a;b]) = UTV^c(-f, [a;b]).$$

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The equality (1) may be viewed as the generalisation of the Hahn-Jordan decomposition of a function with finite total variation.
Moreover, it appears that the function $f^{0,c} : [a;b] \to \mathbb{R}$ defined as

$$f^{0,c}(s) = UTV^c(f,[a;s]) - DTV^c(f,[a;s])$$

has increments differing from the increments of f by no more than c.

The construction of the function f^c

For $g : [a; b] \to \mathbb{R}$ we define its oscillation as

$$\|g\|_{osc} = \sup_{s,u\in[a;b]} |g(s) - g(u)|.$$

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Since for $g^{0,c} = f - f^{0,c}$, $\|g^{0,c}\|_{osc} \le c$, hence for some number α,

$$\sup_{s \in [a;b]} \left|f(s) - f^{0,c}(s) - \alpha\right| = \sup_{s \in [a;b]} \left|g^{0,c}(s) - \alpha\right| \le \frac{1}{2}c \qquad (2)$$

and f^c may be defned as

$$f^c(s) = f^{0,c}(s) + \alpha = UTV^c(f, [a; s]) - DTV^c(f, [a; s]) + \alpha.$$

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$$f^c(s) = f^{0,c}(s) + \alpha = UTV^c(f, [a;s]) - DTV^c(f, [a;s]) + \alpha.$$

Remark

When $\|g^{0,c}\|_{osc} = c$, then the only number α satisfying (2) is

$$\alpha = \inf_{s \in [a;b]} (g^{0,c}(s)) + \frac{1}{2} \|g^{0,c}\|_{osc} = \sup_{s \in [a;b]} (g^{0,c}(s)) - \frac{1}{2} \|g^{0,c}\|_{osc} .$$

Definition of the function $f^{0,c}$ and construction of f^c - problem with adaptation

For any stochastic process $X_t, t \geq 0$, with càdlàg trajectories we may define for every $f = X(\omega)$ the functions $f^{0,c}$ and f^c.

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Observe that from the definition of $f^{0,c}$ it follows that the process $X^{0,c}$, defined pathwise as $X_t^{0,c}(\omega) = f^{0,c}(t)$, where $f = X(\omega)$, is adapted to the natural filtration of the process X, $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$.

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Different observation may be done for analogously defined process X^c. To construct the function f^c one needs to know "future" values of the process X.

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Different observation may be done for analogously defined process X^c. To construct the function f^c one needs to know "future" values of the process X.

Remark

It is relatively easy to construct another process $\tilde{X}^c$, adapted to $\mathcal{F}_t, t \geq 0$, and such that $\left\| X - \tilde{X}^c \right\|_\infty \leq \frac{1}{2}c$ and
$TV\left(\tilde{X}^c, [0;t]\right) \leq TV^c(X, [0;t]) + c/2.$

Definition of the truncated variation, upward truncated variation and downward tuncated variation processes

For arbitrary stochastic process $X_t, t \geq 0$, with càdlàg trajectories, one may define, adapted to the natural filtration

- truncated variation process

$$TV^c(X,t) := TV^c(X,[0;t]);$$

- upward truncated variation process

$$UTV^c(X,t) := UTV^c(X,[0;t])$$

- and downward truncated variation process

$$DTV^c(X,t) := DTV^c(X,[0;t]).$$

Limit distributions of truncated variation processes of Brownian motion with drift as $c \to 0$

Let $X_t = \mu t + B_t$ be a standard Brownian motion with drift μ. Since Brownian motion has infinite total variation, hence for any $t > 0$

$$\lim_{c \downarrow 0} TV^c(X, t) = \infty.$$

It may be interesting (due to the geometric interpretation of truncated variation) to investigate the rate of $TV^c(X, t)$ for small cs. The answer is the following

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Fact

For any $T > 0$ process $c \cdot TV^c(\mu t + B_t, t)$ converges almost surely in $(\mathcal{C}[0; T], \mathbb{R})$ topology to the deterministic function $id : [0; T] \to \mathbb{R}, id(t) = t$.

Generalisation for continuous semimartingales

Let now $X_t, t \geq 0$, be continuous semimartingale, with the decomposition $X_t = X_0 + M_t + A_t$, with M_t being a local martingale and A_t being a continuous process with finite total variation.

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Using the inequality $\max\{|x+y| - c, 0\} \leq \max\{|x| - c, 0\} + |y|$ we obtain, that

$$TV^c(X_t, t) \leq TV^c(M_t, t) + TV^0(A_t, t)$$

and

$$TV^c(M_t, t) \leq TV^c(X_t, t) + TV^0(-A_t, t).$$

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Hence we conclude easily that

$$\lim_{c \downarrow 0} c \cdot TV^c(X, t) = \lim_{c \downarrow 0} c \cdot TV^c(M, t)$$

whenever any of the above limits exist.

We will use the Fact from the previous slide, Dambis and Dubins-Schwarz Theorem saying that every continuous, local martingale $M_t, t \geq 0$, with $M_0 = 0$ and infinite total variation may be represented as

$$M_t = B_{\langle M,M \rangle_t},$$

where B_t is a standard Brownian motion

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$$\lim_{c \downarrow 0} c \cdot TV^c(X, t) = \lim_{c \downarrow 0} c \cdot TV^c(M, t) = \lim_{c \downarrow 0} c \cdot TV^c(B, \langle M, M \rangle_t) = \langle M, M \rangle_t .$$

Generalisation for continuous semimartingales, cont.

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Noticing that $\langle X,X \rangle_t = \langle M,M \rangle_t$, we finally obtain

$$\lim_{c \downarrow 0} c \cdot TV^c(X,t) = \langle X,X \rangle_t .$$

"Second order" convergence for Brownian motion as $c \to 0$

Knowing that for a standard Brownian motion

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It appears that

$$\left(B_t, TV^c(B,t) - \frac{t}{c}\right) \to^d \left(B_t, \frac{\tilde{B}_t}{\sqrt{3}}\right)$$

where the convergence $\to^d$ is understood as the weak convergence in $(\mathcal{C}[0;T], \mathbb{R}^2)$ topology, and $\tilde{B}$ is another standard Brownian motion, independent from B.

"Second order" convergence for Brownian motion with drift as $c \to 0$

Moreover, for standard Brownian motion with drift, $W_t = \mu t + B_t$, we have

$$(W_t, T_t, U_t, D_t) \to^d \left(W_t, \frac{\tilde{B}_t}{\sqrt{3}}, \frac{\tilde{B}_t + \sqrt{3} B_t}{2\sqrt{3}}, \frac{\tilde{B}_t - \sqrt{3} B_t}{2\sqrt{3}} \right) \quad (3)$$

where

$$T_t = TV^c(W, t) - \frac{t}{c},$$

$$U_t = UTV^c(W, t) - \left(\frac{1}{2c} + \frac{\mu}{2} \right) t,$$

$$D_t = DTV^c(W, t) - \left(\frac{1}{2c} - \frac{\mu}{2} \right) t$$

and the convergence $\to^d$ is understood as the weak convergence in $(\mathcal{C}[0; T], \mathbb{R}^4)$ topology and $\tilde{B}$ is again a standard Brownian motion independent from B.

"Second order" convergence for diffusions $c \to 0$

Consider autonomus diffusion process defined as the (unique) strong solution of the sde

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t.$$

For the process X_t and trunceted variation processes of X_t, under the assumption that μ, σ are lipschitz and σ is strictly positive, we have analogous convergence as in (3)

$$\left(X_t, \tilde{T}_t, \tilde{U}_t, \tilde{D}_t\right) \to^d \left(X_t, \frac{\tilde{B}_t}{\sqrt{3}}, \frac{\tilde{B}_t}{2\sqrt{3}}, \frac{\tilde{B}_t}{2\sqrt{3}}\right) \tag{4}$$

where $\tilde{T}_t = TV^c(X,t) - \frac{\langle X \rangle_t}{c}$, $\tilde{U}_t = UTV^c(X,t) - \frac{1}{2}\left(\frac{\langle X \rangle_t}{c} + X_t\right)$, $\tilde{D}_t = DTV^c(X,t) - \frac{1}{2}\left(\frac{\langle X \rangle_t}{c} - X_t\right)$.

The function f^c - construction

In order to define the function f^c we will need some other definitions. For $c > 0$ we define stopping times

$$T_D^c f = \inf\left\{ s \geq a : \sup_{t \in [a;s]} f(t) - f(s) \geq c \right\},$$

$$T_U^c f = \inf\left\{ s \geq a : f(s) - \inf_{t \in [a;s]} f(t) \geq c \right\}.$$

We will assume that $T_D^c f \geq T_U^c f$ i.e. that the first upward jump of the function f of size c appears before the first downward jump of the same size c or both times are infinite, i.e. there is no upward or downward jump of size c.

Note that in the case $T_D^c f < T_U^c f$ we may simply consider the function $-f$.

The function f^c - definition, cont.

Now define sequences $\left(T^c_{U,k}\right)_{k=0}^{\infty}, \left(T^c_{D,k}\right)_{k=-1}^{\infty}$, in the following way: $T^c_{D,-1} = a$, $T^c_{U,0} = T^c_U f$ and for $k = 0, 1, 2, ...$

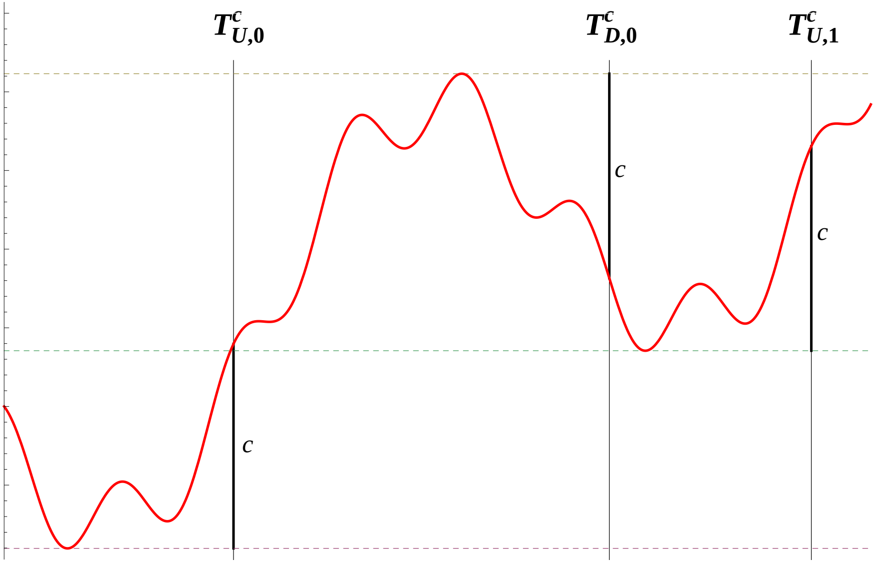
$$T^c_{D,k} = \inf\left\{ s \geq T^c_{U,k} : \sup_{t \in \left[T^c_{U,k}; s\right]} f(t) - f(s) \geq c \right\},$$

$$T^c_{U,k+1} = \inf\left\{ s \geq T^c_{D,k} : f(s) - \inf_{t \in \left[T^c_{D,k}; s\right]} f(t) \geq c \right\}.$$

Remark

Note that there exists such $K < \infty$ that $T^c_{U,K} = \infty$ or $T^c_{D,K} = \infty$. Otherwise we would obtain two infinite sequences $(s_k)_{k=1}^{\infty}, (S_k)_{k=1}^{\infty}$ such that $a \leq s_1 < S_1 < s_2 < S_2 < ... \leq b$ and $f(S_k) - f(s_k) \geq \frac{1}{2}c$. But this is a contradiction, since f is a càdlàg function and $(f(s_k))_{k=1}^{\infty}, (f(S_k))_{k=1}^{\infty}$ have a common limit.

Some pictures



The function f^c - definition, cont.

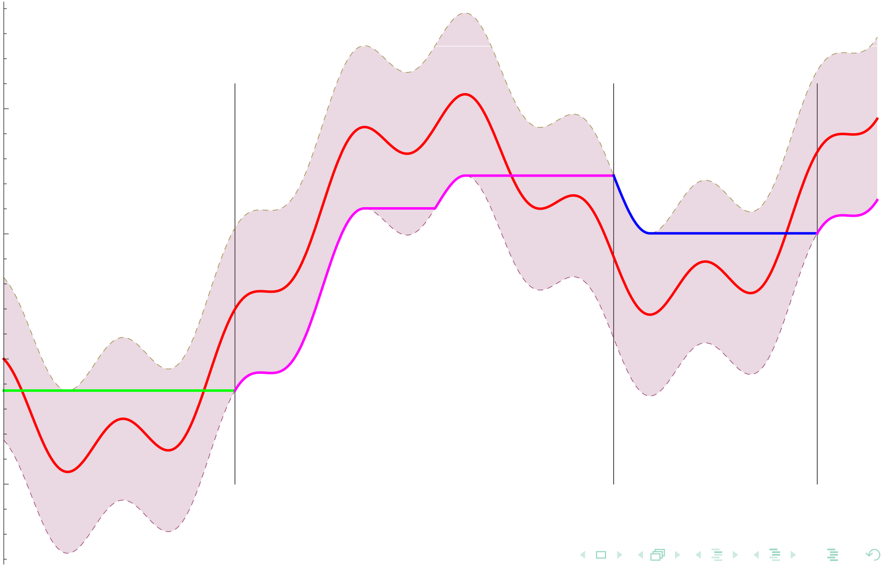
Now let us define two sequences of non-decreasing functions $m_k^c : \left[T_{D,k-1}^c; T_{U,k}^c\right) \to \mathbb{R}$ and $M_k^c : \left[T_{U,k}^c; T_{D,k}^c\right) \to \mathbb{R}$ for such k that $T_{D,k-1}^c < \infty$ and $T_{U,k}^c < \infty$ respectively, with the formulas

$$m_k^c(s) = \inf_{t \in \left[T_{D,k-1}^c; s\right]} f(t), \; M_k^c(s) = \sup_{t \in \left[T_{U,k}^c; s\right]} f(t).$$

The function f^c is now defined with the formulas

$$f^c(s) = \begin{cases} \inf_{t \in \left[0; T_U^c f\right)} f(t) + \frac{1}{2}c \text{ if } s \in \left[a; T_{U,0}^c\right); \\ M_k^c(s) - \frac{1}{2}c \text{ if } s \in \left[T_{U,k}^c; T_{D,k}^c\right), k = 0, 1, 2, ...; \\ m_{k+1}^c(s) + \frac{1}{2}c \text{ if } s \in \left[T_{D,k}^c; T_{U,k+1}^c\right), k = 0, 1, 2, ... \end{cases}$$

Some pictures, cont.



Why the function f^c is optimal?

The function f^c has the smallest total variation possible, since it is monotonic on every interval of the form $\left[T^c_{D,k-1}; T^c_{U,k}\right)$ or $\left[T^c_{U,k}; T^c_{D,k}\right)$, $k = 0, 1, 2, ..., K$, and its variation on these intervals reads as

$$\sup_{t \in \left[T^c_{D,k-1}; T^c_{U,k}\right)} f(t) - \inf_{t \in \left[T^c_{D,k-1}; T^c_{U,k}\right)} f(t) - c$$

or

$$\sup_{t \in \left[T^c_{U,k}; T^c_{D,k}\right)} f(t) - \inf_{t \in \left[T^c_{U,k}; T^c_{D,k}\right)} f(t) - c$$

respectively, thus is the smallest possible for the function from the ball $\left\{g : \|f - g\|_\infty \leq \frac{1}{2}c\right\}$.

Why the function f^c is optimal? - cont.

There is some more accurate reasoning needed if the domain of the function f is not just a sum of intervals of the form $\left[T^c_{D,k-1}; T^c_{U,k}\right)$ and $\left[T^c_{U,k}; T^c_{D,k}\right)$, $k = 0, 1, 2, ..., K$, and it is done with the **minimal** decomposition of the function $f^c - f^c(a)$ into a difference of two non-decreasing functions f^c_U and f^c_D (cf. [L2011b]).
This is possible, since its domain is a sum of the disjoint intervals, where it is monotonic, thus it has finite total variation.
The function f^c_U is constant on the intervals $\left[T^c_{D,k-1}; T^c_{U,k}\right)$ and the function f^c_D is constant on the intervals $\left[T^c_{U,k}; T^c_{D,k}\right)$, $k = 0, 1, 2, ..., K$.
From the very construction of the function f^c it also follows that it belongs to the ball $\left\{g : \|f - g\|_\infty \le \frac{1}{2}c\right\}$ and it is a càdlàg function with jumps possible only in the points where also f has its jumps.

References

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Thank you!