Tail and moment estimates for random chaoses

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Let $X_1, X_2, \ldots$ be a sequence of independent random variables. A random variable of the form

$$S = \sum_{i_1 < i_2 < \ldots < i_d} a_{i_1,\ldots,i_d} X_{i_1} \cdots X_{i_d}$$

is called a (*homogenous, tetrahedral, undecoupled*) *chaos of order* d.

Problem. Find two-sided estimates for moments/tails of S.

Motivation comes from various applications in theory of interated stochastic integrals, random matrices, U-statistics, random graphs etc.

Decoupling inequalities. Let $X_i^{(k)}$, $k = 1, \ldots, d$ be independent copies of X_i and coefficients $a_{\mathbf{i}}$ satisfy the symmetry assumptions i.e. $a_{i_1,\ldots,i_d} = a_{i_{\pi(1)},\ldots,i_{\pi(d)}}$ for any permutation π of $\{1, \ldots, d\}$ and $a_{i_1,\ldots,i_d} = 0$ if $i_j = i_k$ for some $j \neq k$. Define

$$S^{\text{dec}} = \sum_{i_1,i_2,\ldots,i_d} a_{i_1,\ldots,i_d} X_{i_1}^{(1)} \cdots X_{i_d}^{(d)}.$$

Theorem 1 (de la Peña, Montgomery-Smith) *There exists a constant $C(d)$ depending only on the order d such that for any $t > 0$,*

$$C(d)^{-1}\mathbf{P}(|S^{\text{dec}}| \geq C(d)t) \leq \mathbf{P}(|S| \geq t) \leq C(d)\mathbf{P}((|S^{\text{dec}}| \geq t/C(d)).$$

In particular for all $p \geq 1$,

$$C(d)^{-1}\|S^{\text{dec}}\|_p \leq \|S\|_p \leq C(d)\|S^{\text{dec}}\|_p.$$

Nonnegative case I. Let $X_i^{(k)}$ be independent nonnegative random variables with logarithmically concave tails, i.e. the functions $N_i^{(k)}(t) := -\ln \mathbf{P}(X_i^{(k)} > t)$ are convex on $[0,\infty)$ with normalization $N_i^{(k)}(1) = 1$. Suppose that coeficients $a_{\mathbf{i}} = a_{i_1,\ldots,i_d}$ are nonnegative and

$$S := \sum_{\mathbf{i}} a_{\mathbf{i}} X_{i_1}^{(1)} \ldots X_{i_d}^{(d)}.$$

Define

$$B^k_{\mathcal{N},p} := \{x \in \mathbb{R}_n^+ \colon \sum_i N_i^{(k)}(x_i) \le p$$
$$\& x_i = 0 \text{ or } x_i \ge 1 \text{ for all } i\}$$

and for a multiindexed matrix $(a_{\mathbf{i}}) = (a_{i_1,\ldots,i_d})$,

$$\|(a_{\mathbf{i}})\|_{\mathcal{N},p} := \sup\{\sum_{\mathbf{i}} a_{\mathbf{i}} \prod_{k=1}^{d} (1 + x_{i_k}^{(k)}) \colon x^{(k)} \in B^k_{\mathcal{N},p}\}.$$

Theorem 2 (L, Łochowski) *For any $p \ge 1$,*

$$C(d)^{-1}\|(a_{\mathbf{i}})\|_{\mathcal{N},p} \le \|S\|_p \le C(d)\|(a_{\mathbf{i}})\|_{\mathcal{N},p},$$

where $C(d)$ is a constant that depends only on d.

Corollary 1 *For any $t > 0$,*

$$\mathbf{P}(S > C(d)\|(a_{\mathbf{i}})\|_{\mathcal{N},t}) \le e^{-t}$$

and

$$\mathbf{P}(S > c(d)\|(a_{\mathbf{i}})\|_{\mathcal{N},t}) \ge \min(c(d), e^{-t}),$$

where constants $0 < c(d) < C(d) < \infty$ depend only on d.

Example Suppose that X_i are i.i.d. r.v.'s with $Exp(1)$ distribution and $a_{\mathbf{i}}$ satisfy the symmetry assumptions. Then

$$\begin{aligned}\|\sum_{\mathbf{i}} a_{\mathbf{i}} X_{i_1}\cdots X_{i_d}\|_p \\ &\sim_d \sum_{\mathbf{i}} a_{\mathbf{i}} + p \max_{i_1} \sum_{i_2,\ldots,i_d} a_{\mathbf{i}} \\ &\quad + \ldots + p^d \max_{\mathbf{i}} a_{\mathbf{i}}.\end{aligned}$$

Nonnegative Case II. In some applications to random graphs one needs to estimate the tails of chaoses of the form

$$S = \sum_{\mathbf{i}} a_{\mathbf{i}} X_{i_1} \cdots X_{i_d}$$

with $a_{\mathbf{i}} \in \{0, 1\}$ and $\mathbf{P}(X_i = 1) = 1 - \mathbf{P}(X_i = 0) = \alpha$.

If $d = 1$ and $S = \sum_i a_i X_i$, then for $p \geq \ln(1/\alpha)$,

$$\|S\|_p \sim \sup\{\sum_i a_i b_i \colon \sum_i m(b_i) \leq p\},$$

where

$$m(x) := \begin{cases} 0 & \text{if } 0 \leq x \leq e\alpha \\ x \ln(x/(e\alpha)) & \text{if } e\alpha \leq x \leq 1 \\ +\infty & \text{if } x > 1 \end{cases}$$

The precise two-sided estimates are not known in the case $d \geq 2$. Recently Łochowski obtained estimates within $\log^d(1/\alpha)$ factor.

Gaussian chaoses. Let $g_1, g_2, \ldots$ be a sequence of independent standard $\mathcal{N}(0,1)$ normal random variables and

$$S = \sum_{i_1 < i_2 < \ldots < i_d} a_{i_1,\ldots,i_d} g_{i_1} \cdots g_{i_d}$$

be a *Gaussian chaos of order* d.

Case $d = 1$ is trivial, since

$$\|\sum_i a_i g_i\|_p = \Big(\sum_i a_i^2\Big)^{1/2} \|g\|_p \sim \sqrt{p}\Big(\sum_i a_i^2\Big)^{1/2}.$$

For $d = 2$ the estimate was given by Hanson and Wright'71

$$\|\sum_{ij} a_{ij} g_i g_j\|_p \sim \sqrt{p}\|(a_{ij})\|_{\{1,2\}} + p\|(a_{ij})\|_{\{1\}\{2\}},$$

where

$$\|(a_{ij})\|_{\{1,2\}} := \|(a_{ij})\|_{\mathrm{HS}} = (\sum_{ij} a_{ij}^2)^{1/2}$$

and

$$\|(a_{ij})\|_{\{1\}\{2\}} := \|(a_{ij})\|_{l_2\to l_2} = \sup\Big\{\sum_{ij} a_{ij}x_iy_j \colon \|x\|_2 \le 1, \|y\|_2 \le 1\Big\}.$$

For $d \ge 3$ Borell'84 and Arcones and Giné'93 showed that

$$\|S\|_p \sim_d \sum_{k=1}^{d} p^{k/2}\mathbf{E}\sup\Big\{\sum_{\mathbf{i}} a_{\mathbf{i}} \prod_{l=1}^{k} x^{(k)}_{i_k} \prod_{l=k+1}^{d} g^{(l)}_{i_l} \colon \|x^{(l)}\|_2 \le 1, 1 \le l \le k\Big\}.$$

The above formula gives the precise dependence on p, but unfortunately involves suprema of empirical processes that are in general not easy to estimate.

Generalizations to a nongaussian case: Adamczak'05+ and Łochowski'05.

More notation. Let $d \geq 1$ and $A = (a_{\mathbf{i}})_{1 \leq i_1, \ldots, i_d \leq n}$ be a finite multiindexed matrix of order d. If $\mathbf{i} \in \{1, \ldots, n\}^d$ and $I \subset \{1, \ldots, d\}$ then we define $i_I := (i_j)_{j \in I}$. For disjoint nonnempty subsets $I_1, \ldots, I_k$ of $\{1, \ldots, d\}$ we put

$$\|A\|_{I_1 \ldots I_k} := \sup \Big\{ \sum_{\mathbf{i}} a_{\mathbf{i}} x^{(1)}_{i_{I_1}} \cdots x^{(k)}_{i_{I_k}} : \sum_{i_{I_1}} (x^{(1)}_{i_{I_1}})^2 \leq 1, \ldots, \sum_{i_{I_k}} (x^{(k)}_{i_{I_k}})^2 \leq 1 \Big\}.$$

By $S(k, d)$ we denote a set of all partitions of $\{1, \ldots, d\}$ into k nonempty disjoint sets $I_1, \ldots, I_k$. For $p \geq 1$ we put

$$m_p(A) := \sum_{k=1}^{d} p^{k/2} \sum_{(I_1, \ldots, I_k) \in S(k,d)} \|A\|_{I_1 \ldots I_k}.$$

Theorem 3 *For any multiindexed finite matrix* $A = (a_{\mathbf{i}})_{1 \leq i_1, \ldots, i_d \leq n}$ *and* $p \geq 2$ *we have*

$$\frac{1}{C(d)} m_p(A) \leq \Big\| \sum_{\mathbf{i}} a_{\mathbf{i}} \prod_{j=1}^{d} g^{(j)}_{i_j} \Big\|_p \leq C(d) m_p(A).$$

Case $d = 3$. We have for $A = (a_{ijk})$

$$\begin{aligned}\|A\|_{\{1,2,3\}} &= \sup\Big\{\sum a_{ijk}x_{ijk}\colon \sum x_{ijk}^2 \le 1\Big\}\\ &= (\sum a_{ijk}^2)^{1/2},\end{aligned}$$

$$\begin{aligned}\|A\|_{\{1\}\{2,3\}} &= \sup\Big\{\Big(\sum_{jk}\Big(\sum_i a_{ijk}x_i\Big)^2\Big)^{1/2}\colon \sum_i x_i^2 \le 1\Big\}\\ &= \sup\Big\{\Big(\sum_i\Big(\sum_{jk} a_{ijk}y_{jk}\Big)^2\Big)^{1/2}\colon \sum_{jk} y_{jk}^2 \le 1\Big\},\end{aligned}$$

$$\begin{aligned}\|A\|_{\{1\}\{2\}\{3\}} = \sup\Big\{\sum a_{ijk}x_iy_jz_k\colon &\sum_i x_i^2 \le 1,\\ &\sum_j y_j^2 \le 1, \sum_k z_k^2 \le 1\Big\}\end{aligned}$$

and

$$\begin{aligned}m_p(A) = &\sqrt{p}\|A\|_{\{1,2,3\}} + p(\|A\|_{\{1\}\{2,3\}} + \|A\|_{\{1\}\{2,3\}}\\ &+ \|A\|_{\{1\}\{2,3\}}) + p^{3/2}\|A\|_{\{1\}\{2\}\{3\}}.\end{aligned}$$

In particular under the symmetry assumptions

$$m_p(A) = \sqrt{p}\|A\|_{\{1,2,3\}} + p\|A\|_{\{1\}\{2,3\}} + p^{\frac{3}{2}}\|A\|_{\{1\}\{2\}\{3\}}.$$

Tail estimates.

Corollary 2 *For any $t \geq 0$ and $d \geq 1$*

$$\frac{1}{C(d)} \exp\left[-C(d) \min_{1 \leq k \leq d} \min_{(I_1, \ldots, I_k)} \left(\frac{t}{\|A\|_{I_1 \ldots I_k}}\right)^{2/k}\right]$$
$$\leq \mathbf{P}\left(\left|\sum_{\mathbf{i}} a_{\mathbf{i}} g^{(1)}_{i_1} \cdots g^{(d)}_{i_d}\right| \geq t\right)$$
$$\leq C(d) \exp\left[-\frac{1}{C(d)} \min_{1 \leq k \leq d} \min_{(I_1, \ldots, I_k)} \left(\frac{t}{\|A\|_{I_1 \ldots I_k}}\right)^{2/k}\right].$$

In particular for $d = 3$ under the symmetry assumptions,

$$\frac{1}{C} \exp\left[-C \min\left(\left(\frac{t}{\|A\|}\right)^2, \left(\frac{t}{\|A\|}\right), \left(\frac{t}{\|A\|}\right)^{2/3}\right)\right]$$
$$\leq \mathbf{P}\left(\left|\sum_{ijk} a_{ijk} g_i g_j g_k\right| \geq t\right)$$
$$C \exp\left[-\frac{1}{C} \min\left(\left(\frac{t}{\|A\|}\right)^2, \left(\frac{t}{\|A\|}\right), \left(\frac{t}{\|A\|}\right)^{2/3}\right)\right].$$

Few comments about the proof. Lower estimates - easy induction. Upper estimates - induction or Arcones-Giné, but in any case the main problem it to estimate supremum of some Gaussian process - norm of a random correlated multiindexed matrix.

Theorem 4 *For any* $d \geq 2, p \geq 2$,

$$\mathbf{E}\Big\|\Big(\sum_{i_d} a_{\mathbf{i}} g_{i_d}\Big)\Big\|_{\{1\}\ldots\{d-1\}} \leq C(d) p^{(1-d)/2} m_p(A).$$

For $d = 3$ one has a slightly stronger estimate

Theorem 5

$$\mathbf{E}\Big\|\Big(\sum_k a_{ijk} g_k\Big)_{ij}\Big\|_{l_2 \to l_2}$$

$$\leq C \inf_{p \geq 2} \Big(\frac{1}{\sqrt{p}}\|A\|_{\{1,2,3\}} + s(A) + \sqrt{p}\|A\|_{\{1\}\{2\}\{3\}}\Big)$$

$$\leq C\Big(s(A) + \Big(\|A\|_{\{1,2,3\}}\|A\|_{\{1\}\{2\}\{3\}}\Big)^{1/2}\Big),$$

$$s(A) := \|A\|_{\{1\}\{2,3\}} + \|A\|_{\{2\}\{1,3\}}.$$

Other symmetric chaoses. Case $d = 2$. Suppose that $X_i^{(k)}$, $k = 1, 2$ are symmetric r.v. with logarithmically concave tails, i.e. functions $N_i^{(k)}(t) := -\ln \mathbf{P}(|X_i^{(k)}| > t)$ are convex on $[0, \infty)$ with normalization $N_i^{(k)}(1) = 1$. Let

$$M_i^{(k)}(t) := \begin{cases} t^2 & \text{for } |t| \leq 1 \\ N_i(|t|) & \text{for } |t| > 1. \end{cases}$$

For sequences (A_i) and matrices $(a_{i,j})$ and $p > 0$ we put

$$\|(A_i)\|_{\mathcal{N},p}^{(k)} := \sup\{\sum A_i b_i : \sum M_i^{(k)}(b_i) \leq p\}$$

and

$$\|(a_{i,j})\|_{\mathcal{N},\mathcal{N},p} := \sup\{\sum a_{i,j} b_i c_j : \sum M_i^{(1)}(b_i) \leq p, \sum M_j^{(2)}(c_j) \leq p\}.$$

Theorem 6 *For any* $p > 2$

$$\|\sum_{ij} a_{ij} X_i^{(1)} X_j^{(2)}\|_p \sim \|((\sum_j a_{ij}^2)^{1/2})\|_{\mathcal{N},p}^{(1)} + \|((\sum_i a_{ij}^2)^{1/2})\|_{\mathcal{N},p}^{(2)} + \|(a_{i,j})\|_{\mathcal{N},\mathcal{N},p}.$$

Example If ε_i are iid with $\mathbf{P}(\varepsilon_i = \pm 1) = 1/2$ and $S = \sum_{i<j} a_{ij}\varepsilon_i\varepsilon_j$ then under the symmetry assumptions,

$$\|S\|_p \sim \sum_{i \le p} A_i^* + \sqrt{p}(\sum_{i>p} (A_i^*)^2)^{1/2} + \|(a_{ij})\|_{\varepsilon,\varepsilon,p},$$

where A_i^* is the nondecreasing rearrangement of the sequence $((\sum_j a_{ij}^2)^{1/2})$ and

$$\|(a_{ij})\|_{\varepsilon,\varepsilon,p} := \sup\{\sum a_{i,j} b_i c_j \colon \|(b_i)\|_2, \|(c_j)\|_2 \le p, \quad |b_i|, |c_j| \le 1\}.$$

In the case $d > 2$ Łochowski obtained two-sided estimates in terms of suprema of certain empirical processes. He also (up to logarithmic factors) reduces the bounds for general "log-concave" chaoses to the Rademacher ones.

Selected Open Problems.

- Estimates of moments for general nonnegative (symmetric) chaoses of order two.

- Bounds for nonnegative chaoses based on two points random variables.

- Moment estimates for Rademacher chaoses of order 3 and higher.

- Bounds for Hilbert-space valued chaoses.