# Limit theorems for the truncated variation and for numbers of interval crossings of Lévy and self-similar processes

Rafał M. Łochowski and Piotr Miłoś

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**Abstract**

We study the number $\mathrm{n}^{y,c}(X,T)$ of crossings the interval $[y;y+c]$ made by a stochastic process $X_t$, $t\geq 0$, before time $T\geq 0$. In this paper $X$ is either a self-similar process, pure-jump Lévy process with unbounded total variation or a semimartingale with non-zero continuous part. Under mild conditions we establish that, as $c\to 0+$, the quantity $\int_{\mathbb{R}} \mathrm{n}^{y,c}(X,1)\,\mathrm{d}a$ grows as a certain deterministic function $1/\chi(c)$ and, when $X$ is a Lévy process, we obtain asymptotics for the process $\int_{\mathbb{R}} \mathrm{n}^{y,c}(X,t)\,\mathrm{d}a - t/\chi(c)$. For any locally bounded Borel function $g$ we also deal with the limit of the process $\chi(c)\int_{\mathbb{R}} \mathrm{n}^{y,c}(X,\cdot)\,g(a)\mathrm{d}a$. Our main tool are techniques based on the notion of the *truncated variation*, for which we obtain the corresponding results of independent interest.

## 1 Introduction

The classical result, established by Banach [1] and, independently, by Vitali [28], states that for continuous $f$ there exists a close relationship between the total variation of $f$ and the number of level crossings of $f$. The total variation of a real function $f:[a;b]\to\mathbb{R}$ $(-\infty<a<b<+\infty)$ is defined by

$$\mathrm{TV}(f,[a;b]) = \sup_n \sup_{a\leq t_1<t_2<\ldots<t_n\leq b} \sum_{i=1}^{n-1} |f(t_{i+1})-f(t_i)|.$$

Let $N^y(f,[a;b])$, for any $y\in\mathbb{R}$, denote the number of times the graph of $f$ crosses the level $y$ (the precise definition of $N^y(f,[a;b])$ is given in the next section). The classical Banach Indicatrix Theorem states that

$$\mathrm{TV}(f,[a;b]) = \int_{\mathbb{R}} N^y(f,[a;b])\mathrm{d}y. \tag{1}$$

It is also possible to extend this result to the case when $f$ is *regulated*, which means that $f$ has left and right limits, see [23]. Unfortunately, even such result does not tell much, when it is applied to the trajectories of many important stochastic processes, which are characterised by the infinite total variation. However, if $c>0$ and the total variation in (1) is replaced by the *truncated variation*, defined as

$$\mathrm{TV}^c(f,[a;b]) := \sup_n \sup_{a\leq t_1<t_2<\ldots<t_n\leq b} \sum_{i=1}^{n-1} \max\{|f(t_{i+1})-f(t_i)|-c,0\}, \tag{2}$$

and $N^y(f,[a;b])$ is replaced by $\mathrm{n}^{y,c}(f,[a;b])$, denoting the number of times that the function $f$ crosses the interval $[y;y+c]$ (the precise definition of $\mathrm{n}^{y,c}(f,[a;b])$ is stated in the next section), then the following equality holds:

$$\mathrm{TV}^c(f,[a;b]) = \int_{\mathbb{R}} \mathrm{n}^{y,c}(f,[a;b])\,\mathrm{d}y, \tag{3}$$

(see [20, Theorem 1]). Both sides of (3) are *finite* for any regulated $f$. The finiteness of $\mathrm{TV}^c(f,[a;b])$ follows easily from the following variational interpretation of the truncated variation:

$$\mathrm{TV}^c(f,[a;b]) = \inf\left\{\mathrm{TV}(g,[a;b]) : \|f-g\|_{\infty,[a;b]} \le c/2\right\}, \tag{4}$$

where $\|f-g\|_{\infty,[a;b]} = \sup_{t\in[a;b]}|f(t)-g(t))|$, and the fact that the family of regulated functions coincides with the family of uniform limits of step functions (which naturally have finite total variations), see [7, Theorem 7.6.1].

The purpose of this paper is to describe the limit behaviour (as $c \to 0+$) of the the truncated variation process $\mathrm{TV}^c(X,[0;t])$, $t \ge 0$, for self-similar processes, pure-jump Lévy processes and semimartingales with non-zero continuous part. Thanks to relation (3) this way we will also investigate the limit behaviour of the (integrated) numbers of interval crossings for such processes. For Lévy processes these quantities reveal rather strong concentration properties and we will prove that for some deterministic function $\chi : (0,+\infty) \to (0,+\infty)$ the sequence of processes $\mathrm{TV}^c(X,[0;t]) - t/\chi(c)$ has non-trivial limit (almost surely, in the uniform convergence topology on compacts) as $c \to 0+$.

We will be also interested in the limit behaviour of the integrals of the form

$$\chi(c)\int_{\mathbb{R}} \mathrm{n}^{y,c}(X,[0;t])\,g(y)\mathrm{d}y,$$

where $g : \mathbb{R} \to \mathbb{R}$ is some locally bounded, Borel function. This way we will distinguish some of the levels $y$. From the a.s. (resp. in probability) convergence (as $c \to 0+$) of the processes $\chi(c)\mathrm{TV}^c(X,[0;\cdot])$, $c > 0$, to the deterministic limit $\mathrm{Id}(t) = t$, $t \ge 0$, it will follow that $\int_{\mathbb{R}} \chi(c)\mathrm{n}^{y,c}(X,[0;\cdot])\,g(y)\mathrm{d}y$ tends a.s. (resp. in probability) to the limit $\int_0^{\cdot} g(X_s)\,\mathrm{d}s$. This relation resembles the occupation density formula for local times $L_t^y(X)$, $y \in \mathbb{R}, t \ge 0$, of the process $X$:

$$\int_{\mathbb{R}} L_t^y(X)\,g(y)\mathrm{d}y = \int_0^{\cdot} g(X_s)\,\mathrm{d}s$$

and suggests that the limit $\chi(c)\mathrm{n}^{y,c}(X,[0;t])$ exists and may be interpreted as the local time process of $X$, $L^y(X)$. The possibility of the definition of local times as the normalized limit of the number of interval crossings is studied since long time. The answer is positive for continuous semimartingales (El Karoui, [9]) or semimartingales whose jump part has locally finite total variation (see [17]), however when $X$ is a Lévy process with the infinite variation jump part, the situation is more complex. Potential theory of Lévy processes provides easy iff condition, expressed in terms of the characteristic exponent of $X$, when the occupation measure is absolutely continuous with respect to the Lebesgue measure (and in such case this density is the local time [4, Chapt. V, Theorem 1]), but, as far as we know, the possibility of the definition of the local time of $X$ in terms of $\mathrm{n}^{y,c}(X,[0;\cdot])$ is yet to be investigated.

Let us comment on the organisation of the paper. In the next section we establish relationships between the truncated variation and the numbers of interval crossings of a càdlàg process. To deal with the processes of the form $\int_{\mathbb{R}} \mathrm{n}^{y,c}(X,[0;\cdot])\,g(y)\mathrm{d}y$ we will need some results which go beyond (3). To obtain them we will use properties of so called Skorohod map on $[-c/2;c/2]$. Next, we summarize the properties of the truncated variation used in the sequel of this paper. In the third section we deal with the (first order) limit theorems for the truncated variation (when the truncation parameter $c$ tends to 0) of self-similar processes with càdlàg paths (typical examples of such processes are fractional Brownian motions and $\alpha$-stable processes), pure jump Lévy processes and semimartingales with non-zero continuous part. Simultaneously, we will be able to establish corresponding limit theorems for integrated numbers of interval crossings by these processes. In the last, fourth section we will obtain more precise, second order convergence results for the truncated variation of pure jump Lévy processes.

# 2 Definitions and preliminaries

In this section we introduce most of definitions and notation used in the paper. They are stated for deterministic càdlàg functions and in the subsequent sections they will be applied to trajectories of stochastic processes.

## 2.1 Relations between the truncated variation, the numbers of interval crossings and the numbers of level crossings via the Skorohod map

Along with the truncated variation introduced already in (2), we define *upward-* and *downward truncated variations* of a function $f$ on an interval $[a;b]$ by

$$\mathrm{UTV}^c(f,[a;b]) := \sup_n \sup_{a\leq t_1<s_1<t_2<\ldots<t_n<s_n\leq b} \sum_{i=1}^{n-1} \max\{f(s_i)-f(t_i)-c,0\}, \tag{5}$$

and

$$\mathrm{DTV}^c(f,[a;b]) := \sup_n \sup_{a\leq t_1<s_1<t_2<\ldots<t_n<s_n\leq b} \sum_{i=1}^{n-1} \max\{f(t_i)-f(s_i)-c,0\}, \tag{6}$$

respectively. Let us formally define the above–mentioned down– and upcrossings of a càdlàg function.

**Definition 1.** *Let $y\in\mathbb{R}$ and $c>0$ and $[a;b]$ be an interval. The number of times that the function $f:[a;b]\mapsto\mathbb{R}$* upcrosses *the interval $[y;y+c]$ on the interval $[a;b]$ is defined by*

$$\mathrm{u}^{y,c}(f,[a;b]) = \sup_n \sup_{a\leq t_1<s_1<t_2<\ldots<t_n<s_n\leq b} \sum_{i=1}^{n-1} \mathbf{1}_{\{f(t_i)\leq y \text{ and } f(s_i)>y+c\}}.$$

*Further we define the number of* downcrosses *by*

$$\mathrm{d}^{y,c}(f,[a;b]) = \sup_n \sup_{a\leq t_1<s_1<t_2<\ldots<t_n<s_n\leq b} \sum_{i=1}^{n-1} \mathbf{1}_{\{f(s_i)<y \text{ and } f(t_i)\geq y+c\}}.$$

*Finally we set the number of* crosses *by*

$$\mathrm{n}^{y,c}(f,[a;b]) = \mathrm{u}^{y,c}(f,[a;b]) + \mathrm{d}^{y,c}(f,[a;b]).$$

We note that the above quantities are finite for any càdlàg function. They are also decreasing in $c$. We may thus define their analogues for $c=0$ by the limit procedure.

**Definition 2.** *Let $y\in\mathbb{R}$ and $[a;b]$ be an interval. The number of times that the function $f:[a;b]\mapsto\mathbb{R}$* upcrosses *the level $y$ on the interval $[a;b]$ is defined by*

$$\mathrm{u}^{y}(f,[a;b]) = \lim_{c\to 0+} \mathrm{u}^{y,c}(f,[a;b]).$$

*Analogously we define the number of* downcrosses *by*

$$\mathrm{d}^{y}(f,[a;b]) = \lim_{c\to 0+} \mathrm{d}^{y,c}(f,[a;b]).$$

*Finally the number of crosses is given by*

$$\mathrm{n}^{y}(f,[a;b]) = \mathrm{d}^{y}(f,[a;b]) + \mathrm{u}^{y}(f,[a;b]).$$

We stress that these quantities may be infinite.

We will usually work with $f : [0, +\infty) \to \mathbb{R}$ and $[a; b]$ of the form $[0; t]$, $t > 0$. It will be beneficial to have a shorthand notation. To ease notation, for $f : [0; +\infty) \to \mathbb{R}$ and any $t > 0$ we define

$$\mathrm{UTV}^c(f, t) := \mathrm{UTV}^c(f, [0; t]), \quad \mathrm{u}^{y,c}(f, t) = \mathrm{u}^{y,c}(f, [0; t]).$$

Completely analogous definitions will be used for other quantities.

The main purpose of this section is to state a relation between the truncated variation, the numbers of interval crossings of a càdlàg function $f$ and the numbers of level crossings of the regularization of $f$ obtained via so-called double Skorohod map, which will be defined now.

Let $D[0; +\infty)$ denote the set of real-valued càdlàg functions and $BV^+[0; +\infty)$, $BV^0[0; +\infty)$ denote subspaces of $D[0; +\infty)$ consisting of nondecreasing functions and piecewise monotonic functions, respectively.

**Definition 3.** *Given a càdlàg function $f : [0; +\infty) \mapsto \mathbb{R}$ and $c > 0$, a pair of functions $(\phi^{c,x}, \eta^{c,x}) \in D[0; +\infty) \times BV^0[0; +\infty)$ is said to be a solution of the* Skorohod problem on $[-c/2; c/2]$ with starting condition $x \in [-c/2; c/2]$ for $f$ *if the following conditions are satisfied:*

1. *for every $t \geq 0$, $\phi^{c,x}(t) = f(t) + \eta^{c,x}(t) \in [-c/2; c/2]$;*
2. *$\eta^{c,x} = \eta_d^{c,x} - \eta_u^{c,x}$, where $\eta_d^{c,x}, \eta_u^{c,x} \in BV^+[0; +\infty)$ and the corresponding measures $\mathrm{d}\eta_d^{c,x}$, $\mathrm{d}\eta_u^{c,x}$ are supported by $\{t \geq 0 : \phi^{c,x}(t) = -c/2\}$ and $\{t \geq 0 : \phi^{c,x}(t) = c/2\}$ respectively;*
3. *$\phi^{c,x}(0) = x$.*

Existence and uniqueness of the solution of this problem (even in more general setting, with the interval $[-c/2; c/2]$ replaced by a time-varying interval $[\alpha(t); \beta(t)]$) is given in [21, Proposition 2.7]). A very similar problem, with the starting condition $\phi^x(0) = x$ replaced by the condition $\phi(0-) = 0$ is well-known in the literature, see for example [6], [15]. From now on, for any càdlàg $f$ we implicitly denote

$$f^{c,x} = -\eta^{c,x},$$

where $\eta^{c,x}$ fulfills Definition 3.

Let us now summarize properties of the Skorohod mapping (proofs are contained in [21, Proposition 2.9 and inequalities (2.1), (2.2) and (2.3)]).

**Proposition 4.** *Let $f : [0; +\infty) \to \mathbb{R}$ be càdlàg, $c > 0$ and $x \in [-c/2; c/2]$. Then*

*(A)* $\|f - f^{c,x}\|_\infty \leq c/2$;

*(B)* $f^{c,x}(0) = f(0) - x$;

*(C)* *$f^{c,x}$ is càdlàg and has finite total variation on any finite interval;*

*(D)* *for any $t \geq 0$, the jumps $\Delta f(t) := f(t) - f(t-)$ and $\Delta f^{c,x}(t) := f^{c,x}(t) - f^{c,x}(t-)$ satisfy*

$$|\Delta f^{c,x}(t)| \leq |\Delta f(t)| \quad \textit{and} \quad \Delta f^{c,x}(t)\Delta f(t) \geq 0;$$

*(E)* *for any $t \geq 0$,*

$$\mathrm{TV}^c(f, t) \leq \mathrm{TV}(f^{c,x}, t) \leq \mathrm{TV}^c(f, t) + c;$$

*(F)* *analogously, for any $t \geq 0$,*

$$\mathrm{UTV}^c(f, t) \leq \mathrm{UTV}(f^{c,x}, t) \leq \mathrm{UTV}^c(f, t) + c$$

*and*

$$\mathrm{DTV}^c(f, t) \leq \mathrm{DTV}(f^{c,x}, t) \leq \mathrm{DTV}^c(f, t) + c.$$

The Jordan decomposition implies

$$\mathrm{TV}(f^{c,x},t)=\mathrm{UTV}(f^{c,x},t)+\mathrm{DTV}(f^{c,x},t) \tag{7}$$

and

$$f^{c,x}(t)=\mathrm{UTV}(f^{c,x},t)-\mathrm{DTV}(f^{c,x},t)\,. \tag{8}$$

Using (8) and relations (A) and (F) we get

$$f(t)-\frac{3}{2}c\le \mathrm{UTV}^c(f,t)-\mathrm{DTV}^c(f,t)\le f(t)+\frac{3}{2}c. \tag{9}$$

Next, from [18, Theorem 4] we have

$$\mathrm{UTV}^c(f,t)+\mathrm{DTV}^c(f,t)=\mathrm{TV}^c(f,t)\,. \tag{10}$$

Using (9) and (10) we can obtain, with the accuracy proportional to $c$, estimates of $\mathrm{TV}^c(f,t)$ , $\mathrm{UTV}^c(f,t)$ , $\mathrm{DTV}^c(f,t)$ or $f$, whenever two other of these quantities are known, for example:

$$2\cdot\mathrm{UTV}^c(f,t)-f(t)-\frac{3}{2}c\le \mathrm{TV}^c(f,t)\le 2\cdot\mathrm{UTV}^c(f,t)-f(t)+\frac{3}{2}c. \tag{11}$$

From [21, Lemma 3.3, Lemma 3.4] and Definition 1 we immediately get the following result, linking the interval crossings of a càdlàg function $f$ and the level crossings of its regularisation $f^{c,x}$.

**Lemma 5.** *Let $f:[0;+\infty)\to\mathbb{R}$ be càdlàg and let for $c>0$, $x\in[-c/2;c/2]$, $f^{c,x}=-\eta^{c,x}$, where $(\phi^{c,x},\eta^{c,x})$ is the solution of the Skorohod problem on $[-c/2;c/2]$ with the starting condition $x$ for $f$. For any $y\in\mathbb{R}$ we have*

$$\mathrm{d}^{y,c}(f,t)\le \mathrm{d}^{y}(f^{c,x},t)\le \mathrm{d}^{y,c}(f,t)+1,$$

$$\mathrm{u}^{y,c}(f,t)\le \mathrm{u}^{y}(f^{c,x},t)\le \mathrm{u}^{y,c}(f,t)+1.$$

*Moreover, taking $x=c/2$ we get $\mathrm{d}^{y,c}(f,t)=\mathrm{d}^{y}(f^{c,x},t)$ and taking $x=-c/2$ we get $\mathrm{u}^{y,c}(f,t)=\mathrm{u}^{y}(f^{c,x},t)$ .*

Now we state a relation of the upward (resp. downward) truncated variation of $f\in BV^0[0;+\infty)$ with the integrated number of upcrossings (resp. downcrossings).

**Lemma 6.** *Let $f\in BV^0[0;+\infty)$, $t>0$ and $g:\mathbb{R}\mapsto\mathbb{R}$ be locally bounded and Borel–measurable. Then*

$$\begin{aligned}\int_{\mathbb{R}} g(a)\mathrm{u}^a(f,t)\,\mathrm{d}a=&\int_0^t g(f(s-))\mathrm{UTV}(f,\mathrm{d}s)\\&+\sum_{0<s\le t,\Delta f(s)>0}\int_{f(s-)}^{f(s)}[g(a)-g(f(s-))]\mathrm{d}a,\end{aligned} \tag{12}$$

*and analogously*

$$\begin{aligned}\int_{\mathbb{R}} g(a)\mathrm{d}^a(f,t)\,\mathrm{d}a=&\int_0^t g(f(s-))\mathrm{DTV}(f,\mathrm{d}s)\\&+\sum_{0<s\le t,\Delta f(s)<0}\int_{f(s)}^{f(s-)}[g(a)-g(f(s-))]\mathrm{d}a.\end{aligned} \tag{13}$$

*Proof.* By assumption there exists a finite sequence of intervals $\{I_i\}$, which are mutually disjoint, $\bigcup_i I_i = [0,t]$ and the function is monotone on any of $I_i$. Given a level $a$ with any interval $I_i$ we can have at most one associated upcrossing. This happens when

$$a \in [m_i, M_i), \quad \text{where } m_i = \inf_{x\in \bar{I}_i} f(x), M_i = \sup_{x\in \bar{I}_i} f(x).$$

Clearly, neither of intervals where function is decreasing, induces any upcrossing. Let $J$ be a subset on the indices where the function is increasing. Then we have

$$\mathrm{u}^a(f,t) = \sum_{i\in J} \mathbf{1}_{\{a\in[m_i,M_i)\}}.$$

Consequently

$$\int_{\mathbb{R}} g(a)\mathrm{u}^a(f,t)\,\mathrm{d}a = \sum_{i\in J} \int_{m_i}^{M_i} g(a)\mathrm{d}a.$$

We are now to deal with the integral. To this end we apply the classical idea of opening temporal windows at times of jumps. We may impose that the sum of the lengths of these windows is finite and consider interpolated continuous $\tilde{f}$. We have

$$\int_{m_i}^{M_i} g(a)\mathrm{d}a = \int_{\tilde{f}(s)}^{\tilde{f}(t)} g(a)\mathrm{d}a,$$

for properly defined $s, t$. Then we apply the change of variable for the Riemann-Stieltjes integral

$$\int_{\tilde{f}(s)}^{\tilde{f}(t)} g(a)\mathrm{d}a = \int_s^t g\left(\tilde{f}(u)\right)\mathrm{d}\tilde{f}\,(u)\,.$$

Let us consider the decomposition of the measure $\mathrm{UTV}(f,\mathrm{d}s)$ on the interval $I_i$ into the continuous part $\mu_c$ and the atomic part $\mu_a$. Moreover, let $T \subset [s,t]$ be the set of the temporal windows. Clearly we have

$$\int_s^t \mathbf{1}_{u\notin T} g\left(\tilde{f}(u)\right)\mathrm{d}\tilde{f}\,(u) = \int_{I_i} g(f(s-))\mu_c(\mathrm{d}s),$$

moreover

$$\int_s^t \mathbf{1}_{u\in T} g\left(\tilde{f}(u)\right)\mathrm{d}\tilde{f}\,(u) = \sum_{s\in \bar{I}_i, \Delta f(s)>0} \int_{f(s-)}^{f(s)} g(a)\mathrm{d}a.$$

Using this (12) holds by simple calculations.

In a similar way one proves (13). □

## 2.2 Limit theorem for the integrated numbers of interval crossings with respect to the measure $g(a)\mathrm{d}a$

Using Lemma 6 we will prove the following theorem, relating the integrated number of interval crossings of a function $f : [0;+\infty) \to \mathbb{R}$ with the asymptotics of the truncated variation $\mathrm{TV}^c(f,t)$, $t \geq 0$, as $c \to 0+$.

**Theorem 1.** *Let $f : [0;+\infty) \to \mathbb{R}$ be a càdlàg function. Assume that there exist a function $\varphi : (0;+\infty) \to [0;+\infty)$, such that $\lim_{c\to 0+} \varphi(c) = 0$, and a function $\zeta : [0;+\infty) \to \mathbb{R}$, such that for $t \geq 0$:*

$$\varphi(c)\mathrm{TV}^c(f,t) \to \zeta(t) \text{ as } c \to 0+\,. \tag{14}$$

*Then $\zeta$ is non–negative, non–decreasing, continuous and for any continuous function $g : \mathbb{R} \to \mathbb{R}$ we have the following convergences*

$$\varphi(c)\int_{\mathbb{R}} \mathrm{n}^{a,c}(f,\cdot)\,g(a)\mathrm{d}a \to \int_0^{\cdot} g(f(t-))\zeta(\mathrm{d}t),$$

$$\varphi(c)\int_{\mathbb{R}}\mathrm{u}^{a,c}(f,\cdot)\,g(a)\mathrm{d}a \to \frac{1}{2}\int_0^{\cdot} g(f(t-))\zeta(\mathrm{d}t),$$

*and*

$$\varphi(c)\int_{\mathbb{R}}\mathrm{d}^{a,c}(f,\cdot)\,g(a)\mathrm{d}a \to \frac{1}{2}\int_0^{\cdot} g(f(t-))\zeta(\mathrm{d}t),$$

*in the uniform convergence topology on compact subsets of positive half-line* $[0;+\infty)$.

**Remark 7.** *By (11) and (9) the convergence in (14) is equivalent to any of*

$$\varphi(c)\mathrm{UTV}^c(f,t) \to \frac{1}{2}\zeta(t) \text{ as } c \to 0+ \tag{15}$$

*or*

$$\varphi(c)\mathrm{DTV}^c(f,t) \to \frac{1}{2}\zeta(t) \text{ as } c \to 0+.$$

**Remark 8.** *Theorem 1 remains valid when the convergence $c \to 0+$ is replaced by a convergence along some sequence $c_n \to 0+$ as $n \to +\infty$. To see this it is enough to replace everywhere in the proof $c$ by $c_n$.*

*Proof.* Assume that (15) holds. The fact that $\zeta$ is non-negative and non-decreasing follows immediately from the same properties of the function $t \mapsto \mathrm{TV}^c(f,t)$, $t \geq 0$. The fact that $\zeta$ is continuous follows from Proposition 9 (property (J)), proven in the next subsection, and the fact that $\lim_{c\to 0+}\varphi(c) = 0$. Now let us fix $T > 0$ and put

$$M = \max\left\{\sup_{0\leq t\leq T}|f(t)|, \sup_{0\leq t\leq T}|\zeta(t)| = \zeta(T)\right\}.$$

Using the assumption that $g$ is continuous, for fixed $\varepsilon > 0$ we can find a number $\delta > 0$ such that

$$\sup_{x,y\in[-M-1;M+1],|x-y|\leq\delta}|g(x)-g(y)| \leq \frac{\varepsilon}{M}. \tag{16}$$

Let us also define

$$N = \#\{t \in (0;T] : |f(t) - f(t-)| > \delta\}. \tag{17}$$

Let $t \in [0;T]$. We use Lemma 5 with $x = -c/2$ and Lemma 6 to obtain

$$\begin{aligned}\int_{\mathbb{R}} g(a)\mathrm{u}^{a,c}(f,t)\,\mathrm{d}a &= \int_{\mathbb{R}} g(a)\mathrm{u}^{a}(f^{c,x},t)\,\mathrm{d}a \\ &= \int_0^t g(f^{c,x}(s-))\mathrm{UTV}(f^{c,x},\mathrm{d}s) \\ &\quad + \sum_{0<s\leq t,\Delta f^{c,x}(s)>\delta}\int_{f^{c,x}(s-)}^{f^{c,x}(s)}[g(a)-g(f^{c,x}(s-))]\mathrm{d}a \\ &\quad + \sum_{0<s\leq t,0<\Delta f^{c,x}(s)\leq\delta}\int_{f^{c,x}(s-)}^{f^{c,x}(s)}[g(a)-g(f^{c,x}(s-))]\mathrm{d}a. \end{aligned}\tag{18}$$

Let us denote the consecutive summands on the right side of equation (18) by $L_1(c)$, $L_2(c)$ and $L_3(c)$ respectively.

First, for $c > 0$ we estimate

$$\varphi(c)L_2(c) \leq \varphi(c)\cdot 2N(2M+c)\left(2\sup_{x\in[-M-c/2;M+c/2]}|g(x)|\right) \to 0 \quad \text{as } c \to 0+. \tag{19}$$

Using (16) and then Proposition 4 (property (F)) we estimate

$$\begin{aligned}\varphi(c)L_3(c) &\leq \varphi(c)\sum_{0<s\leq t,0<\Delta f^{c,x}(s)\leq\delta}|\Delta f^{c,x}(s)|\sup_{x,y\in[-M-c/2;M+c/2],|x-y|\leq\delta}|g(x)-g(y)| \\ &\leq \varphi(c)\mathrm{UTV}(f^{c,x},t)\frac{\varepsilon}{M} \leq \varphi(c)(\mathrm{UTV}^c(f,t)+c)\frac{\varepsilon}{M} \leq 2\varepsilon \end{aligned}\tag{20}$$

for $c$ small enough such that $\varphi(c)\mathrm{UTV}^c(f,T)+\varphi(c)c \le 2M$.

Now we are left with $L_1$. We fix $K=1,2,\ldots$ and define

$$I_i = \left[-M+2M\frac{i-1}{K}; -M+2M\frac{i}{K}\right], \quad i=\{1,2,...,K\}.$$

Further we define the following sequence $\{\upsilon_k\}_{k\ge 0}$ of times. Let $\upsilon_0=0$ and

$$\upsilon_k = \inf\left\{t>\upsilon_{k-1} : f(t)\in I_i \text{ for some } i=1,2,...,K \text{ such that } f(\upsilon_{k-1})\notin I_i\right\}.$$

We have

$$\begin{aligned} L_1(c) =& \sum_{k=0}^{+\infty}\int_{(\upsilon_k\wedge t;\upsilon_{k+1}\wedge t]} g(f^{c,x}(s-))\mathrm{UTV}(f^{c,x},\mathrm{d}s) \\ =& \sum_{k=0}^{+\infty}\int_{(\upsilon_k\wedge t;\upsilon_{k+1}\wedge t]} \left\{g(f^{c,x}(s-))-g(f(\upsilon_k\wedge t))\right\}\mathrm{UTV}(f^{c,x},\mathrm{d}s) \\ &+\sum_{k=0}^{+\infty} g(f(\upsilon_k\wedge t))\mathrm{UTV}(f^{c,x},(\upsilon_k\wedge t;\upsilon_{k+1}\wedge t]). \end{aligned} \tag{21}$$

Using (16) it is easy to estimate for $c\le 2\delta$ and $2M/K\le\delta$ the first summand in (21) multiplied by $\varphi(c)$ :

$$\begin{aligned} &\varphi(c)\sum_{k=0}^{+\infty}\int_{(\upsilon_k\wedge t;\upsilon_{k+1}\wedge t]} \left|g(f^{c,x}(s-))-g(f(s-))+g(f(s-))-g(f(\upsilon_k\wedge t))\right|\mathrm{UTV}(f^{c,x},\mathrm{d}s) \\ &\le \varphi(c)\sum_{k=0}^{+\infty}\frac{2\varepsilon}{M}\mathrm{UTV}(f^{c,x},(\upsilon_k\wedge t;\upsilon_{k+1}\wedge t]) \le \varphi(c)\,\mathrm{UTV}(f^{c,x},t)\,\frac{2\varepsilon}{M} \\ &\le \varphi(c)\left(\mathrm{UTV}^c(f,t)+c\right)\frac{2\varepsilon}{M}\le 4\varepsilon \end{aligned} \tag{22}$$

for $c$ small enough such that $\varphi(c)\mathrm{UTV}^c(f,T)+\varphi(c)c\le 2M$. Now we investigate the second summand of $L_1$ multiplied by $\varphi(c)$ :

$$\begin{aligned} &\sum_{k=0}^{+\infty} g(f(\upsilon_k\wedge t))\varphi(c)\mathrm{UTV}(f^{c,x},(\upsilon_k\wedge t;\upsilon_{k+1}\wedge t]) \\ &\to \frac{1}{2}\sum_{k=0}^{+\infty} g(f(\upsilon_k\wedge t))\left\{\zeta(\upsilon_{k+1}\wedge t)-\zeta(\upsilon_k\wedge t)\right\} \quad \text{as } c\to 0+. \end{aligned} \tag{23}$$

Since for each $t\in[0;T]$ there is only finite number of $k=0,1,\ldots$ for which $\upsilon_k\le t$, the convergence in (23) is uniform on $[0;T]$. Moreover for $2M/K\le\delta$

$$\begin{aligned} &\left|\int_{(0;t]} g(f(s-))\zeta(\mathrm{d}s)-\sum_{k=0}^{+\infty} g(f(\upsilon_k\wedge t))\left\{\zeta(\upsilon_{k+1}\wedge t)-\zeta(\upsilon_k\wedge t)\right\}\right| \\ &=\left|\sum_{k=0}^{+\infty}\int_{(\upsilon_k\wedge t;\upsilon_{k+1}\wedge t]}\left\{g(f(s-))-g(f(\upsilon_k\wedge t))\right\}\zeta(\mathrm{d}s)\right| \\ &\le \sum_{k=0}^{+\infty}\int_{(\upsilon_k\wedge t;\upsilon_{k+1}\wedge t]}\frac{\varepsilon}{M}\zeta(\mathrm{d}s)=\frac{\varepsilon}{M}\zeta(t)\le\frac{\varepsilon}{M}\zeta(T)\le\varepsilon. \end{aligned} \tag{24}$$

From (18)-(24) we get

$$\sup_{0\le t\le T}\left|\varphi(c)\int_{\mathbb{R}} g(a)\mathrm{u}^{a,c}(f,t)\,\mathrm{d}a-\frac{1}{2}\int_{(0;t]} g(f(s-))\zeta(\mathrm{d}s)\right|\le 10\varepsilon$$

for any $c>0$ sufficiently close to 0.

The proof of the convergence of $\int_{\mathbb{R}} g(a)\mathrm{d}^{a,c}(f,t)\,\mathrm{d}a$ is analogous. The convergence of $\int_{\mathbb{R}} g(a)\mathrm{n}^{a,c}(f,t)\,\mathrm{d}a$ follows from the convergences of $\int_{\mathbb{R}} g(a)\mathrm{u}^{a,c}(f,t)\,\mathrm{d}a$ and $\int_{\mathbb{R}} g(a)\mathrm{d}^{a,c}(f,t)\,\mathrm{d}a$ and the relation $\mathrm{n}^{a,c}(f,t)=\mathrm{u}^{a,c}(f,t)+\mathrm{d}^{a,c}(f,t)$. □

## 2.3 Properties of the truncated variation

The following proposition summarises the properties of the truncated variation (most of them are used in the paper).

**Proposition 9.** *Let $f : [a;b] \to \mathbb{R}$ be a càdlàg function on the interval $[a;b]$ and $c > 0$. Then the following properties hold.*

*(A) For any strictly increasing and continuous function $s : \mathbb{R} \supset A \to \mathbb{R}$ such that $[a;b] \subset Im(s) = s(A)$,*

$$\mathrm{TV}^c(f, [a;b]) = \mathrm{TV}^c\big(f \circ s, \left[s^{-1}(a); s^{-1}(b)\right]\big). \tag{25}$$

*(B) We have*

$$\mathrm{TV}^c(f, [a;b]) = \mathrm{TV}^c(-f, [a;b]) \tag{26}$$

*and*

$$\mathrm{UTV}^c(f, [a;b]) = \mathrm{DTV}^c(-f, [a;b]). \tag{27}$$

*(C) Moreover,*

$$\mathrm{UTV}^c(f, [a;b]) + \mathrm{DTV}^c(f, [a;b]) = \mathrm{TV}^c(f, [a;b]). \tag{28}$$

*(D) For any $s \in (a;b)$ we have*

$$\mathrm{TV}^c(f, [a;b]) \geq \mathrm{TV}^c(f, [a;s]) + \mathrm{TV}^c(f, [s;b]) \tag{29}$$

*and the analogous inequalities hold for $\mathrm{UTV}^c$ and $\mathrm{DTV}^c$.*

*(E) On the other hand for any $s \in (a;b)$,*

$$\mathrm{TV}^c(f, [a;b]) \leq \mathrm{TV}^c(f, [a;s]) + \mathrm{TV}^c(f, [s;b]) + c \tag{30}$$

*and the analogous inequalities hold for $\mathrm{UTV}^c$ and $\mathrm{DTV}^c$.*

*(F) For any $c_1, c_2 > 0$ we have*

$$\mathrm{TV}^{c_1}(f, [a;b]) = \frac{c_1}{c_2}\mathrm{TV}^{c_2}\left(\frac{c_2}{c_1}f, [a;b]\right) \tag{31}$$

*and the analogous relations hold for $\mathrm{UTV}^c$ and $\mathrm{DTV}^c$.*

*(G) For any $g : [a;b] \to \mathbb{R}$ and $c_1, c_2 \geq 0$ we have*

$$\mathrm{TV}^{c_1+c_2}(f + g, [a;b]) \leq \mathrm{TV}^{c_1}(f, [a;b]) + \mathrm{TV}^{c_2}(g, [a;b]) \tag{32}$$

*and the analogous inequalities hold for $\mathrm{UTV}^c$ and $\mathrm{DTV}^c$. In (32) we admit some quantities to be infinite in case $c_1 = 0$ or $c_2 = 0$.*

*(H) For any $g : [a;b] \to \mathbb{R}$ and $c > 0$ we have*

$$|\mathrm{TV}^c(f + g, [a;b]) - \mathrm{TV}^c(f, [a;b])| \leq \mathrm{TV}(g, [a;b]) \tag{33}$$

*and the analogous inequalities hold for $\mathrm{UTV}^c$ and $\mathrm{DTV}^c$.*

*(I) We have*

$$\lim_{c \to 0+} \mathrm{TV}^c(f, [a;b]) = \mathrm{TV}(f, [a;b]) \tag{34}$$

*and the analogous convergences hold for $\mathrm{UTV}^c$ and $\mathrm{DTV}^c$. Recall that the right-hand side might be infinite.*

*(J) The function $[a,b] \ni s \mapsto V(s) := \mathrm{TV}^c(f, [a;s])$ is càdlàg and for any $s \in (a;b)$ we have*

$$|f(s) - f(s-)| \geq V(s) - V(s-) \geq \max\{|f(s) - f(s-)| - c, 0\}. \tag{35}$$

*(K) Let $g : [a;b] \to \mathbb{R}$ be càdlàg and $x \in (a,b)$. Let us assume that $|f(x) - f(x-)| = |g(x) - g(x-)| \geq c$ and $f(y) - f(x) = g(y) - g(x)$ for all $y \in [x;b]$. Then*

$$\mathrm{TV}^c(f,[a;y]) - \mathrm{TV}^c(f,[a;z]) = \mathrm{TV}^c(g,[a;y]) - \mathrm{TV}^c(g,[a;z]) \quad \textit{for any} \quad y \geq z \geq x. \tag{36}$$

*The same holds for* $\mathrm{UTV}^c$ *and* $\mathrm{DTV}^c$.

*Proof.* The properties (A), (B), (D), (E), (F), (G), (H), (I) are easy. (C) can be found in [18, Theorem 4.1]. We now prove (J). The first inequality follows from definition (2) and the elementary inequality

$$(|f(u) - f(s)| - c)_+ \leq (|f(u) - f(s-)| - c)_+ + |f(s) - f(s-)|,$$

for any $u \in [a;s)$. The inequality $V(s) - V(s-) \geq (|f(s) - f(s-)| - c)_+$ follows from the superadditivity property (29). One may for example define the function $\tilde{f} : [a;b] \to \mathbb{R}$, $\tilde{f}(u) = f(u)$ for $u \in [a;s)$, $\tilde{f}(u) = f(s-)$ for $u \in [s;b)$ and $\tilde{f}(b) = f(s)$. It is easy to see that $V(s) = \mathrm{TV}^c\left(\tilde{f},[a;b]\right) \geq \mathrm{TV}^c\left(\tilde{f},[a;s]\right) + \mathrm{TV}^c\left(\tilde{f},[s;b]\right) = V(s-) + (|f(s) - f(s-)| - c)_+$.

The proof of (K) requires more effort. To avoid formal problems we assume that the functions are defined on $[0;+\infty)$, we leave to the reader transferring the proof to $[a;b]$. First, we recall a renewal-like structure associated with the truncated variation. We define the following times: $T^c_{D,-1} = 0$ and for $k = 0,1,\ldots$ we put

$$T^c_{U,k}f := \inf\left\{ s \geq T^c_{D,k-1} : f(s) - \inf_{T^c_{D,k-1} \leq t \leq s} f(t) \geq c \right\}, \tag{37}$$

$$T^c_{D,k}f := \inf\left\{ s \geq T^c_{U,k} : \sup_{T^c_{U,k} \leq t \leq s} f(t) - f(s) \geq c \right\}. \tag{38}$$

By [18, Theorem 4.1]) we have

$$\mathrm{UTV}^c(f,[0;s]) = \sum_{k=0}^{K(s)} (M_k - m_k - c) + (M_{K(s)+1}(s) - m_{K(s)+1} - c)\mathbf{1}_{\left\{s \in \left[T^c_{U,K(s)+1}f; T^c_{D,K(s)+1}f\right)\right\}}, \tag{39}$$

where $K(s) = i$ if $s \in [T^c_{D,i}f, T^c_{D,i+1}f)$ (and $K(s) = -1$ if $s < T^c_{D,0}f$) and $M_k, m_k$ for $k$ such that $T^c_{D,k-1}f < +\infty$, $T^c_{U,k}f < +\infty$ respectively are defined in the following way (see also [18, p. 127–128])

$$m^c_k = \inf_{t \in \left[T^c_{D,k-1}f; T^c_{U,k}f\right)} f(t), \tag{40}$$

$$M^c_k = \sup_{t \in \left[T^c_{U,k}f; T^c_{D,k}f\right)} f(t). \tag{41}$$

$M_{K(s)}(s)$ is defined as

$$M^c_{K(s)}(s) = \sup_{t \in \left[T^c_{U,K(s)+1}f; s\right]} f(t). \tag{42}$$

Notice that $M_k, m_k$ and $M_{K(s)}(s)$ increase by $d$ if a constant $d$ is added to the function $f$. Clearly we have

$$\begin{aligned}\Delta := \mathrm{UTV}^c(f,[0;y]) - \mathrm{UTV}^c(f,[0;z]) = \sum_{k=K(z)+1}^{K(y)} (M_k - m_k - c) \\ +(M_{K(y)+1}(y) - m_{K(y)+1} - c)\mathbf{1}_{\left\{y \in \left[T^c_{U,K(y)+1}f; T^c_{D,K(y)+1}f\right)\right\}} \\ -(M_{K(z)+1}(z) - m_{K(z)+1} - c)\mathbf{1}_{\left\{z \in \left[T^c_{U,K(z)+1}f; T^c_{D,K(z)+1}f\right)\right\}}.\end{aligned} \tag{43}$$

An easy case is when $T^c_{D,K(z)} \geq x$, then the quantities in (43) depend only on the increments of the function on the interval $[x;+\infty)$. Assume otherwise that $T^c_{D,K(z)} < x$. In this case, from the definition of $K(z)$ it must be that $T^c_{U,K(z)+1} = x$, indeed the jump at $x$ cannot be downward. Further, since $z \geq x$, we have $z \in [T^c_{U,K(z)+1};+\infty)$ and from (43) we see that $\Delta$ depends only on the increments of $f$ on $[x;+\infty)$. In this way we have proven the property asserted in (K) for $\mathrm{UTV}^c$. A very similar argument holds for $\mathrm{DTV}^c$ and consequently by the property (C) also holds for $\mathrm{TV}^c$.

□

# 3 LLN for the truncated variation and for the integrated numbers of level crossings of self-similar processes, Lévy processes and semimartingales

We shall introduce the stochastic setting for the following part of the paper. From now on, we make the standing assumption that **any stochastic process $X_t$, $t \geq 0$, considered in this paper attains real values, has càdlàg paths and is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t, t \geq 0), \mathbb{P})$ for which the usual conditions hold**.

Motivated by Theorem 1 and the variational interpretation (4) of the truncated variation, in this section we present the first order limit theorems for the truncated variation of self-similar processes with stationary increments, for Lévy processes without a Brownian component as well as for semimartingales with non-zero continuous part. For a process $X$ from one of the mentioned families we will look for a proper normalisation $\varphi(c)$ such that for each $T > 0$ the family of functions $[0;T] \ni t \mapsto \varphi(c)\mathrm{TV}^c(X(\omega),t)$, $c > 0$, has (in a sense defined later) a non-trivial limit as $c \to 0+$. It appears that for semimartingales the continuous part (if it is not equal zero) dominates the limit behaviour of the truncated variation and the proper normalisation of the truncated variation to obtain a non-trivial limit has smaller order (as $c \to 0+$) then for pure-jump Lévy processes. Simultaneously, using Theorem 1 we will establish limit theorems for integrated numbers of interval crossings by the mentioned processes.

Let us introduce some notation for various notions of convergence of stochastic processes. Let $\{Y^c\}_{c>0}$ be a family of càdlàg stochastic processes on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t, t \geq 0), \mathbb{P})$.

- '$Y^c \to Y$' will denote the almost sure convergence of $Y^c$ to a process $Y$ as $c \to 0+$ in the topology of uniform convergence on compacts, i.e. for any $T > 0$, $\sup_{0\leq t\leq T} |Y^c_t - Y_t| \to 0$ a.s. as $c \to 0+$.
- '$Y^c \to^{\mathbb{P}} Y$' will denote the convergence in probability ($\mathbb{P}$) of $Y^c$ to a process $Y$ as $c \to 0+$ in the topology of uniform convergence on compacts.
- '$Y^c \to^d Y$' will denote the convergence of $Y^c$ to a process $Y$ as $c \to 0+$ in the sense of finite dimensional distributions.
- '$Y^c \implies Y$' will denote the weak functional convergence of $Y^c$ to a process $Y$ as $c \to 0+$ in the Skorohod J topology on compacts, i.e. for any any $T > 0$ and any bounded and continuous function $F : \mathcal{D}([0;T]) \to \mathbb{R}$, we have $\mathbb{E}F(Y^c_t, 0 \leq t \leq T) \to \mathbb{E}F(Y_t, 0 \leq t \leq T)$, as $c \to 0+$, where $\mathcal{D}([0;T])$ is the space of càdlàg functions $f : [0;T] \mapsto \mathbb{R}$ equipped with the Skorohod J topology (for the definition see for example [27, Chapt. 1]).

Directly from (11) and (9) we have.

**Proposition 10.** *Let "$\longrightarrow$" denote any of the stated modes of convergence. Let $X_t$, $t \geq 0$, be a càdlàg process and let us assume that there exists a function $\varphi : (0;+\infty) \to [0;+\infty)$*

*such that* $\lim_{c\to 0+}\varphi(c)=0$ *and a process* $\zeta_t$, $t\ge 0$, *such that:*

$$\varphi(c)\mathrm{TV}^c(X,\cdot)\longrightarrow\zeta \text{ as } c\to 0+. \tag{44}$$

*Then convergence (44) is equivalent to any of*

$$\varphi(c)\mathrm{UTV}^c(X,\cdot)\longrightarrow\frac{1}{2}\zeta \text{ as } c\to 0+$$

*or*

$$\varphi(c)\mathrm{DTV}^c(X,\cdot)\longrightarrow\frac{1}{2}\zeta \text{ as } c\to 0+.$$

Next, from Theorem 1 we have

**Proposition 11.** *Let* $X_t$, $t\ge 0$, *be a càdlàg process and let us assume that there exists a function* $\varphi:(0;+\infty)\to[0;+\infty)$ *such that* $\lim_{c\to 0+}\varphi(c)=0$ *and a process* $\zeta_t$, $t\ge 0$, *such that:*

$$\varphi(c)\mathrm{TV}^c(X,\cdot)\to\zeta \text{ as } c\to 0+. \tag{45}$$

*Then* $\zeta$ *is non–negative, non–decreasing, continuous and for any continuous function* $g:\mathbb{R}\to\mathbb{R}$ *we have*

$$\varphi(c)\int_{\mathbb{R}}\mathrm{n}^{a,c}(f,\cdot)\,g(a)\mathrm{d}a\to\int_0^{\cdot}g(f(t-))\zeta(\mathrm{d}t). \tag{46}$$

*Similarly, if*

$$\varphi(c)\mathrm{TV}^c(X,\cdot)\to^{\mathbb{P}}\zeta \text{ as } c\to 0+ \tag{47}$$

*then* $\zeta$ *is non–negative, non–decreasing, continuous and for any continuous function* $g:\mathbb{R}\to\mathbb{R}$ *we have*

$$\varphi(c)\int_{\mathbb{R}}\mathrm{n}^{a,c}(f,\cdot)\,g(a)\mathrm{d}a\to^{\mathbb{P}}\int_0^{\cdot}g(f(t-))\zeta(\mathrm{d}t). \tag{48}$$

*Finally, similar convergences hold for* $\varphi(c)\int_{\mathbb{R}}\mathrm{u}^{a,c}(f,\cdot)\,g(a)\mathrm{d}a$ *and* $\varphi(c)\int_{\mathbb{R}}\mathrm{d}^{a,c}(f,\cdot)\,g(a)\mathrm{d}a$.

*Proof.* The implication (45) $\Rightarrow$ (46) follows directly from Theorem 1.

To prove the implication (47) $\Rightarrow$ (48) let us recall the following subsequence criterion (cf. [13, Lemma 4.2] ): a sequence of random variables $Y_n$, $n=1,2,\ldots$, converges in probability to a random variable $Y_\infty$ if and only if from any strictly increasing sequence $n_k$, $k=1,2,\ldots$, of positive integers it is possible to choose a subsequence $n_{k_l}$, $l=1,2,\ldots$, such that $Y_{n_{k_l}}$, $l=1,2,\ldots$, converges almost surely to $Y_\infty$. Let us choose any increasing sequence $n_k$, $k=1,2,\ldots$, of positive integers. Assuming that (47) holds and using the subsequence criterion we find a subsequence $n_{k_l}$, $l=1,2,\ldots$, such that

$$\varphi(c_{n_{k_l}})\,TV^{c_{n_{k_l}}}(X,\cdot)\to\zeta \text{ as } l\to+\infty.$$

From this and Remark 8 it follows that

$$\varphi(c_{n_{k_l}})\int_{\mathbb{R}}n^{a,c_{n_{k_l}}}(f,\cdot)\,g(a)\mathrm{d}a\to\int_0^{\cdot}g(f(t-))\zeta(\mathrm{d}t).$$

This means that the subsequence criterion hods for the family of processes $\varphi(c)\int_{\mathbb{R}}n^{a,c}(f,\cdot)\,g(a)\mathrm{d}a$, $c>0$, and we obtain the desired convergence in probability. □

In this and the following sections by Id we will denote the identity process (i.e. $\mathrm{Id}(t)=t$ for $t\ge 0$).

## 3.1 LLN for the truncated variation of self-similar processes with stationary increments

Let us start with recalling that a process $(X_t, t \geq 0)$ is called self-similar with index $\beta > 0$ if for any $A > 0$

$$\left(A^{-\beta} X_{At}, t \geq 0\right) =^d (X_t, t \geq 0), \tag{49}$$

where $=^d$ denotes the equality of distributions.

A stochastic process $(X_t, t \geq 0)$ has stationary increments if for any $0 \leq t_1 < t_2 \ldots < t_n$ and any $h > 0$ we have

$$\begin{aligned}&(X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \ldots X_{t_n} - X_{t_{n-1}})\\ &=^d (X_{t_2+h} - X_{t_1+h}, X_{t_3+h} - X_{t_2+h}, \ldots X_{t_n+h} - X_{t_{n-1}+h}).\end{aligned}$$

**Theorem 2.** *Let $X$ be a self-similar process with index $\beta > 0$, which has stationary increments. We assume additionally that for some $c_0 > 0$ it fulfills*

$$\mathbb{E}\mathrm{TV}^{c_0}(X, [0;1]) < +\infty \tag{50}$$

*and that it has trivial invariant $\sigma$-field.Then for some $C > 0$ the following convergence holds*

$$c^{1/\beta - 1}\mathrm{TV}^c(X, \cdot) \to^{\mathbb{P}} C \cdot \mathrm{Id}.$$

**Remark 12.** *In view of* (11) *condition* (62) *is equivalent to* $\mathbb{E}\mathrm{UTV}^c(X, [0,1])_0 < +\infty$ *and* $\mathbb{E}|X_1| < +\infty$*, which, due to (9), is equivalent to* $\mathbb{E}\mathrm{DTV}^c(X, [0,1])_0 < +\infty$ *and* $\mathbb{E}|X_1| < +\infty$.

**Remark 13.** *The main drawback of Theorem 2 is that it does not identify the constant $C$.*

*Proof.* By (49), (31) and (25) we get

$$\begin{aligned}&c^{1/\beta - 1}\mathrm{TV}^c(X_s, [0;t]) =^d c^{1/\beta - 1}\mathrm{TV}^c(cX_{c^{-1/\beta}s}, [0;t])\\ &= c^{1/\beta}\mathrm{TV}^1(X_{c^{-1/\beta}s}, [0;t]) = c^{1/\beta}\mathrm{TV}^1(X_s, [0; c^{-1/\beta}t]),\end{aligned}$$

where the last two equalities hold on the process level. We define the family $\{\eta(n,m)\}_{m>n\geq 0}$ of random variables

$$\eta(n,m) := \mathrm{TV}^1(X, [n;m]).$$

The truncated variation depends only on the increments, hence $\eta(n+1, m+1) =^d T \circ \eta(n,m)$, where $T$ is the shift operator associated with the process of stationary increments. By (29) we have

$$\eta(0,n) \geq \eta(0,m) + \eta(m,n).$$

Recalling (62) and applying Kingman's subadditive ergodic theorem, see for example [14] or [8, Theorem 6.1], we obtain

$$\lim_{n\to+\infty} \frac{\eta(0,n)}{n} = Z \ a.s.,$$

where $Z \in (-\infty, +\infty]$ is a random variable measurable with respect to the invariant $\sigma$-field, which, by assumption, is trivial thus $Z = \mathit{const}$ almost surely. Further, from (30) we know that

$$\eta(0,n) \leq \sum_{i=0}^{n-1} \eta(i, i+1) + n,$$

hence we get $Z < +\infty$. The convergence in probability in the uniform convergence topology on compacts follows now from the fact that $t \mapsto \mathrm{TV}^c(X, [0;t])$ is non-decreasing. □

A standard example of a self-similar process with stationary increments where Theorem 2 may be applied is fractional Brownian motion. The fractional Brownian motion $B_t^H$, $t \geq 0$ with the Hurst parameter $H \in (0;1)$, is a centered Gaussian process which is self-similar (with index $\beta = H$) and has stationary increments. More precisely, the difference $B_t^H - B_s^H$, $s, t \geq 0$, has normal distribution with the variance $(t-s)^{2H}$.

Other examples of self-similar processes provide strictly $\alpha-$stable processes ($\alpha \in (0;2]$). We will comment more on this case in the next subsection.

### 3.2 LLN for the truncated variation of Lévy processes and for semimartingales with a non-zero continuous part

When $X$ is a Lévy process with a Brownian component then it is a semimartingale with non-zero continuous part. When $X$ is a pure-jump Lévy process without a Brownian part then the order of proper normalisation $\varphi(c)$ at 0 (such that for any $t > 0$, $\varphi(c)\mathrm{TV}^c(X,t)$ converges almost surely as $c \to 0+$) is greater than in the case of a semimartingale with non-zero continuous part. The exact order of this normalisation in terms of the Lévy measure may be calculated as follows.

Let $X_t$, $t \geq 0$, be a real Lévy process with the characteristic exponent given by

$$\Psi(\theta) := \log \mathbb{E} e^{\mathrm{i}\theta X_t} = a\mathrm{i}\theta + \int_{\mathbb{R}\setminus\{0\}} \left(e^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x 1_{|x|\leq 1}\right) \nu(\mathrm{d}x), \tag{51}$$

where $a$ is the drift term and $\nu$ is the Lévy measure. Let the measure $\mu$ of a Borel set $B \subset (0;+\infty)$ be given by $\mu(B) = \nu(B) + \nu(-B)$.

**Theorem 3.** *Let $X_t$, $t \geq 0$, be a Lévy process with the characteristic exponent* (51)*. Then there exists a deterministic function $f : (0;+\infty) \mapsto (0;+\infty)$ such that*

$$(f(c))^{-1}\,\mathrm{TV}^c(X,\cdot) \to \mathrm{Id}. \tag{52}$$

*Moreover, there exist constants $C_1, C_2 > 0$ such that for $c$ small enough we have*

$$C_1 h(c) \leq f(c) \leq C_2 h(c), \tag{53}$$

*where*

$$h(c) := \int_{[2c;1]} x\mu(\mathrm{d}x) + c^{-1}\int_{(0;2c)} x^2\mu(\mathrm{d}x).$$

Theorem 3 follows immedialtely from much stronger result - Theorem 11.

Now we turn to the case when $X$ is a semimartingale with non-zero continuous part. In this case the proper normalisation is simply given by $\varphi(c) = c$ and we have the following result.

**Theorem 4.** *Let $X_t$, $t \geq 0$, be a càdlàg semimartingale and let $\langle X\rangle^{cont}$ denote the continuous part of the quadratic variation of $X$. Then the following convergence holds*

$$c \cdot \mathrm{TV}^c(X,\cdot) \to \langle X\rangle^{cont}.$$

Theorem 4 is a generalisation of [22, Theorem 1] and it is proven in [19], see [19, Theorem 2]. From Theorem 4 it follows that for a Lévy process $X$ without the Brownian component one has $c \cdot \mathrm{TV}^c(X,\cdot) \to 0$, thus the normalisation $(f(c))^{-1}$ used in Theorem 3 must be of greater order as $c \to 0+$ than $c$.

From Theorem 4 immediately we get

**Corollary 14.** *Let $X_t$, $t \geq 0$, be a Lévy process with the characteristic exponent given by*

$$\Psi(\theta) = a\mathrm{i}\theta + \sigma^2\frac{\theta^2}{2} + \int_{\mathbb{R}\setminus\{0\}} \left(e^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x 1_{|x|\leq 1}\right) \nu(\mathrm{d}x),$$

*where $a$ is the drift term, $\sigma \neq 0$ and $\nu$ is the Lévy measure. Then the following convergence holds*

$$c \cdot \mathrm{TV}^c(X,\cdot) \to \sigma^2\mathrm{Id}.$$

### 3.2.1 LLN for the truncated variation of strictly $\alpha-$stable processes

Strictly $\alpha-$stable processes are Lévy processes with the characteristic exponent

$$\Psi(\theta) = \begin{cases} \tilde{C}_0|\theta|^\alpha \left(1 - \mathrm{i}\gamma \tan\frac{\pi\alpha}{2}\mathrm{sgn}\theta\right) & \text{for } \alpha \in (0;1)\cup(1;2], \\ \tilde{C}_0|\theta| + \mathrm{i}\eta\theta & \text{for } \alpha = 1, \end{cases} \tag{54}$$

where $\tilde{C}_0 > 0$ is the scale parameter and $\gamma \in [-1;1]$ is the skewness parameter. The Lévy measure of a strictly $\alpha-$stable process with $\alpha \in (0;2)$ is given by

$$\Pi(\mathrm{d}x) = \begin{cases} \tilde{C}_1 x^{-\alpha-1} & \text{for } x > 0, \\ \tilde{C}_2(-x)^{-\alpha-1} & \text{for } x < 0, \end{cases}$$

where $\tilde{C}_1, \tilde{C}_2 \geq 0$, $\tilde{C}_1 + \tilde{C}_2 > 0$. In the case $\alpha = 1$ we necessarily have $\tilde{C}_1 = \tilde{C}_2$. Strictly $\alpha-$stable process $X_t$, $t \geq 0$, is self-similar process with index $\beta = 1/\alpha$ and the increment $X_t - X_s$, $t > s \geq 0$, has a strictly $\alpha-$stable distribution (cf. [16, Sect. 1.2.6], [29]).

From Theorem 3 it follows that for strictly $\alpha-$stable processes with $\alpha \in (1;2)$ the proper normalisation is $\varphi(c) = (f(c))^{-1} = O\left(c^{\alpha-1}\right)$ and for $\alpha = 1$, $\varphi(c) = (f(c))^{-1} = O\left(1/\ln(1/c)\right)$ as $c \to 0+$ . For $\alpha \in (0;1)$ we naturally have that $\lim_{c\to 0+} \mathrm{TV}^c(X,t) = \mathrm{TV}(X,t) < +\infty$ a.s. for any $t > 0$, thus $\varphi(c) \equiv 1$. In fact, for strictly $\alpha-$stable processes with $\alpha \in (1;2)$ we may also use Theorem 2 and we obtain the following result.

**Proposition 15.** *Let $\alpha \in (1;2)$ and $X_t$, $t \geq 0$, be a strictly $\alpha-$stable process. Then the following limit exists*

$$A := \lim_{N\to+\infty} \frac{\mathbb{E}\mathrm{TV}^1(X,[0;N])}{N} \tag{55}$$

*and*

$$c^{\alpha-1}\mathrm{TV}^c(X,\cdot) \to A\cdot \mathrm{Id}.$$

*Proof.* We easily check that the assumptions of Theorem 2 are satisfied. From the proof of Theorem 2 it follows that the constant $A$ is properly defined, i.e. the limit (55) exists. Moreover, we have the convergence

$$c^{\alpha-1}\mathrm{TV}^c(X,\cdot) \to^{\mathbb{P}} A\cdot \mathrm{Id}.$$

On the other hand, from Theorem 3 we get that this convergence must hold almost surely. □

We can not use Theorem 2 in the case $\alpha \in (0;1]$ since the condition $\mathbb{E}\mathrm{TV}^{c_0}(X,[0;1]) < +\infty$ for some $c_0 > 0$ is violated. However, in the case $\alpha = 1$ we have

**Proposition 16.** *If $X_t$, $t \geq 0$, is a strictly $1-$stable process with the characteristic exponent given by (54), then*

$$\frac{1}{\ln(1/c)}\mathrm{TV}^c(X,\cdot) \to \frac{2}{\pi}\tilde{C}_0\cdot \mathrm{Id}. \tag{56}$$

Proposition 16 follows immediately from a stronger central limit theorem for the truncated variation of strictly $1-$stable processes (Proposition 25) proven in Section 4.

### 3.2.2 Reneval structure of the truncated variation of Lévy processes

Using the renewal-like structure of the (upward)truncated variation given in equation (39) we are able to state a theorem which gives some interpretation of the normalising function $\varphi(c)$ for Lévy processes. Moreover, this theorem encompasses both situations - when the Brownian component exists or not.

Let $X_t$, $t \geq 0$, be a Lévy process. We define stopping times $T^c_{U,k}$ and $T^c_{D,k}$, $k = 0, 1, \ldots$, in the following way. For fixed $\omega \in \Omega$ we set $f = X(\omega)$ and

$$T^c_{U,k}(\omega) := T^c_{U,k} f, \quad T^c_{D,k}(\omega) := T^c_{D,k} f, \quad k = 0, 1, \ldots,$$

where times $T^c_{U,k} f$, $T^c_{D,k} f$, $k = 0, 1, \ldots$, are given by (37) and (38) respectively. Now we define

$$\theta^c_U := \mathbb{E} T^c_{D,0},$$

$$\xi^c_U := \left( \sup_{T^c_{U,0} \leq t < T^c_{D,0}} X_t - \inf_{0 \leq s < T^c_{U,0}} X_s - c \right)_+ = \sup_{0 \leq s < t < T^c_{D,0}} (X_t - X_s - c)_+$$

and

$$\eta^c_U := \mathbb{E} \xi^c_U.$$

**Theorem 5.** *Let $X_t$, $t \geq 0$, be a Lévy process with infinite total variation on any interval $[0; t]$, $t > 0$. Assume that*

$$\mathbb{E} \sup_{0 \leq t < T^{c_0}_{D,0}} X_t < +\infty \tag{57}$$

*for some $c_0 > 0$ and define $\chi_U(c) := \theta^c_U / \eta^c_U$. If*

$$\text{for any } u > 0, \frac{\mathbb{P}(\xi^c_U \leq u/\chi_U(c))}{\theta^c_U} \to 1 \text{ as } c \to 0+ \tag{58}$$

*then we have the following convergence*

$$\chi_U(c) \mathrm{UTV}^c(X, \cdot) \to \mathrm{Id}.$$

**Remark 17.** *The assumptions of Theorem 5 are rather strong. Notice that from the stochastic continuity and the assumption that $X$ is no monotonic on any interval it follows $\xi^c_U \to^{\mathbb{P}} 0$ as $c \to 0+$. Moreover, since $X$ has no finite variation on any interval, to obtain the convergence we must have $\chi_U(c) \to 0$ as $c \to 0+$. Unfortunately, we do not know if the condition $\mathbb{P}(\xi^c_U \leq u/\chi_U(c)) / \theta^c_U \to 1$ for any $u > 0$ as $c \to 0+$ follows e.g. from (57).*

**Remark 18.** *Theorem 5 is closely related to [26, Proposition 3.14] which establishes the convergence of the number $N^c$ of leaves of a Lévy process tree trimmed on level $c$. In our notation it states that $(\theta^c_U + \theta^c_D) N^c \to^{\mathbb{P}} 1$ as $c \to 0$ (the quantity $\theta^c_D$ is defined as $\theta^c_U$ for the process $-X$). The Lebesgue measure of the trimmed tree defined by $L^c = \int_c^{+\infty} N^b \mathrm{d}b$ differs from $\frac{1}{2}\mathrm{TV}^c(X, 1)$ no more then $\sup_{0 \leq t \leq 1} X_t - \inf_{0 \leq t \leq 1} X_t$. Our result thus gives $\chi_U(c) L^c \to 2$.*

*Proof.* First we will prove weak convergence. The proof hinges upon the renewal-like structure of the truncated variation process. Using relation (39) we have

$$\mathrm{UTV}^c\big(X, T^c_{D,k}\big) =^d \sum_{0 \leq i \leq k} \xi^c_{U,k}, \tag{59}$$

where $\xi^c_{U,k}$, $k = 1, 2, \ldots$ are independent copies of $\xi^c_U$.

We easily check that processes $(\kappa^c_U, t \geq 0)$ defined for $c > 0$ by

$$\kappa^c_U(t) := \sum_{0 \leq k \leq \lfloor t/\theta^c_U \rfloor} \left( T^c_{D,k} - T^c_{D,k-1} \right),$$

where $T^c_{D,-1} := 0$, converge weakly (in the Skorohod J topology) to the process Id. Indeed the variable $\kappa^c_{U,0} := T^c_{D,0}$ satisfies conditions $(\mathcal{S}_4)$ and $(\mathcal{S}_5)$ (with $n_c = 1/\theta^c_U$) of [27, p.

283] as $c \to 0+$ (see [27, Subsection 4.5.2, pp. 283, 284]). Conditions $(\mathcal{S}_4)$ and $(\mathcal{S}_5)$ (with $\pi_1 \equiv 0$ and $c(u) \equiv 1$ in the notation of [27]) for the variable $\kappa^c_{U,0}$ read respectively as

$$\frac{\mathbb{P}\left(\kappa^c_{U,0} > u\right)}{\theta^c_U} \to 0$$

and

$$\frac{\mathbb{E}\left[\kappa^c_{U,0}; \kappa^c_{U,0} \le u\right]}{\theta^c_U} = \frac{\mathbb{E}\left[\kappa^c_{U,0}; \kappa^c_{U,0} \le u\right]}{\mathbb{E}\kappa^c_{U,0}} \to 1$$

for any $u > 0$ as $c \to 0+$. In our case the second condition is fulfilled by the fact that $\theta^c_U \to 0$ as $c \to 0+$ and $\kappa^c_{U,0}/(2\theta^c_U)$ is stochastically dominated by a geometric variable (see [26, proof of Proposition 3.14]) while the first condition follows from the second condition and the Chebyschev inequality

$$0 \le \frac{\mathbb{P}\left(\kappa^c_{U,0} > u\right)}{\theta^c_U} \le \frac{\mathbb{E}\left[\kappa^c_{U,0}; \kappa^c_{U,0} > u\right]}{u \cdot \mathbb{E}\kappa^c_{U,0}} \to 0.$$

Now, for any $c > 0$ we define a process $\nu^c_U(t), t \ge 0$, by

$$\nu^c_U(t) := \sup\left\{s : \kappa^c_U(s) \le t\right\}.$$

Clearly

$$\sum_{0 \le k \le \lfloor \nu^c_U(t)/\theta^c_U \rfloor - 1} \left(T^c_{D,k} - T^c_{D,k-1}\right) \le t \le \sum_{0 \le k \le \lfloor \nu^c_U(t)/\theta^c_U \rfloor} \left(T^c_{D,k} - T^c_{D,k-1}\right).$$

We also define a process $\xi_U(t)$, $t \ge 0$ by

$$\xi^c_U(t) = \sum_{0 \le k \le \lfloor t/\theta^c_U \rfloor - 1} \xi^c_{U,k}.$$

By (59) we get

$$\xi^c_U(\nu^c_U(t)) = \sum_{0 \le k \le \lfloor \nu^c_U(t)/\theta^c_U \rfloor - 1} \xi^c_{U,k} \le \mathrm{UTV}^c(X, t) \le \sum_{0 \le k \le \lfloor \nu^c_U(t)/\theta^c_U \rfloor} \xi^c_{U,k}. \tag{60}$$

By assumption (58), for any $u > 0$ we have

$$\frac{\mathbb{P}\left(\chi_U\left(c\right)\xi^c_U > u\right)}{\theta^c_U} \to 0$$

as $c \to 0+$. Thus the variable $\chi_U(c)\,\xi^c_U$ satisfies conditions $(\mathcal{S}_4)$ and $(\mathcal{S}_5)$ (with $n_c = 1/\theta^c_U$) of [27, p. 283] and by [27, Subsection 4.5.2, pp. 283, 284] we get that the process $\chi_U(c)\,\xi^c_U(\cdot)$ converges weakly (in the Skorohod J topology) to the process Id.

Now, we notice that the assumptions of [27, Theorem 4.5.5, p. 290] are satisfied in our case:

- $(\mathcal{T}_4)$ (see [27, p. 287]) assumes that $\left(T^c_{D,k} - T^c_{D,k-1}, \xi^c_{U,k}\right)$, $k = 0, 1, \dots$ are i.i.d. random variables;
- $(\mathcal{A}_{64})$ (see [27, p. 283]) states that the process $\kappa^c_U(\cdot)$ converges weakly in the Skorohod J topology;
- $(\mathcal{A}_{65})$ (see [27, p. 288]) states that the process $\chi_U(c)\,\xi^c_U(\cdot)$ converges weakly in the Skorohod J topology;

- $(\mathcal{I}_{20})$ (see [27, p. 285]) states that that conditions $(\mathcal{S}_4)$ and $(\mathcal{S}_5)$ for $\kappa^c_{D,0}$ hold with $c > 0$ or $\pi_1(0+) = +\infty$ (in the notation of [27]) which in our case holds since in our case (in the notation of [27]) $c \equiv 1$.

Thus we get that

$$\chi_U(c)\,\xi^c_U(\nu^c_D(\cdot)) \Longrightarrow \mathrm{Id}.$$

From this, using (60) we get the claimed weak convergence for $\mathrm{UTV}^c(X,\cdot)$. Now, since the normalising function is deterministic, the a.s. convergence in the uniform convergence topology on compacts follows from Theorem 3. □

The main drawback of the previous result is that the scaling function $\chi$ is implicit. The next theorem gives explicit result for "almost $\alpha$-stable" spectrally negative Lévy process.

**Theorem 6.** *Let $X$ be a Lévy process without Brownian component, with the Lévy measure $\nu$ such that*

$$\nu(\mathrm{d}x) = \frac{L(x)}{(-x)^{1+\alpha}} 1_{x<0}\mathrm{d}x \tag{61}$$

*for $\alpha \in (1;2)$ and some Borel-measurable function $L : (-\infty;0) \mapsto [0;+\infty)$, slowly varying at 0. Then for $C_\alpha = \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)}$*

$$\frac{c^{\alpha-1}}{L(-c)}\mathrm{UTV}^c(X,\cdot) \to C_\alpha \cdot \mathrm{Id}.$$

Theorem 6 follows directly from Theorem 5 and calculations of the normalising function $\chi_U(c)$ for the the spectrally negative process $X$. This may be done with the help of some recent results in the theory of fluctuations of spectrally negative Lévy processes, see [24]. Since the calculations are rather involved and specific for spectrally assymetric Lévy processes we present them in the Appendix.

## 3.3 Related results for the integrated numbers of level crossings

Now, using Proposition 11, we may formulate the counterparts of the results stated in the preceding subsections, expressed in terms of interval crossings. For any continuous function $g : \mathbb{R} \to \mathbb{R}$ we have.

**Theorem 7.** *Let $X$ be a self-similar process with index $\beta > 0$, which has stationary increments. We assume additionally that for some $c_0 > 0$ it fulfills*

$$\mathbb{E}\mathrm{TV}^{c_0}(X,[0;1]) < +\infty \tag{62}$$

*and that it has trivial invariant $\sigma$-field. Then there exists $C > 0$ such that the following convergence holds*

$$c^{1/\beta-1}\int_{\mathbb{R}} \mathrm{n}^{y,c}(X,\cdot)\,g(y)\mathrm{d}y \to^{\mathbb{P}} C\int_0^{\cdot} g(X_{t-})\mathrm{d}t.$$

**Theorem 8.** *Let $X_t$, $t \geq 0$, be a Lévy process with the characteristic exponent (51). Then there exists a deterministic function $f : (0;+\infty) \mapsto (0;+\infty)$ such that*

$$(f(c))^{-1}\int_{\mathbb{R}} \mathrm{n}^{y,c}(X,\cdot)\,g(y)\mathrm{d}y \to \int_0^{\cdot} g(X_{t-})\mathrm{d}t.$$

*Moreover, there exist constants $C_1, C_2 > 0$ such that for $c$ small enough we have*

$$C_1 h(c) \leq f(c) \leq C_2 h(c), \tag{63}$$

*where*

$$h(c) := \int_{[2c;1]} x\mu(\mathrm{d}x) + c^{-1}\int_{(0;2c)} x^2\mu(\mathrm{d}x).$$

**Theorem 9.** *Let $X_t$, $t \geq 0$, be a càdlàg semimartingale and let $\langle X \rangle^{cont}$ denote the continuous part of the quadratic variation of $X$. Then the following convergence holds*

$$c \int_{\mathbb{R}} \mathrm{n}^{y,c}(X, \cdot)\, g(y) \mathrm{d}y \to \int_0^{\cdot} g(X_{t-}) \langle X \rangle_t^{cont}.$$

**Corollary 19.** *Let $X_t$, $t \geq 0$, be a Lévy process with the characteristic exponent given by*

$$\Psi(\theta) = a\mathrm{i}\theta + \sigma^2 \frac{\theta^2}{2} + \int_{\mathbb{R}\setminus\{0\}} \left( \mathrm{e}^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x 1_{|x|\leq 1} \right) \nu(\mathrm{d}x),$$

*where $a$ is the drift term, $\sigma \neq 0$ and $\nu$ is the Lévy measure. Then the following convergence holds*

$$c \int_{\mathbb{R}} \mathrm{n}^{y,c}(X, \cdot)\, g(y) \mathrm{d}y \to \sigma^2 \int_0^{\cdot} g(X_{t-}) \mathrm{d}t.$$

**Proposition 20.** *Let $\alpha \in (1;2)$ and $X_t$, $t \geq 0$, be a strictly $\alpha-$stable process. Then the following limit exists*

$$A := \lim_{N \to +\infty} \frac{\mathbb{E}\mathrm{TV}^1(X, [0;N])}{N} \tag{64}$$

*and*

$$c^{\alpha-1} \int_{\mathbb{R}} \mathrm{n}^{y,c}(X, \cdot)\, g(y) \mathrm{d}y \to A \int_0^{\cdot} g(X_{t-}) \mathrm{d}t.$$

**Proposition 21.** *If $X_t$, $t \geq 0$, is a strictly $1-$stable process with the characteristic exponent given by (54), then*

$$\frac{1}{\ln(1/c)} \int_{\mathbb{R}} \mathrm{n}^{y,c}(X, \cdot)\, g(y) \mathrm{d}y \to \frac{2}{\pi} \tilde{C}_0 \int_0^{\cdot} g(X_{t-}) \mathrm{d}t.$$

**Theorem 10.** *Let $X$ be a Lévy process without Brownian component, with the Lévy measure $\nu$ such that*

$$\nu(\mathrm{d}x) = \frac{L(x)}{(-x)^{1+\alpha}} 1_{x<0} \mathrm{d}x$$

*for $\alpha \in (1;2)$ and some Bore function $L : (-\infty;0) \mapsto [0;+\infty)$, slowly varying at 0. Then for $C_\alpha = \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)}$*

$$\frac{c^{\alpha-1}}{L(-c)} \int_{\mathbb{R}} \mathrm{u}^{y,c}(X, \cdot)\, g(y) \mathrm{d}y \to C_\alpha \int_0^{\cdot} g(X_{t-}) \mathrm{d}t.$$

## 3.4 A remark on local times

One of the most common definition of the local times $L = L_t^y$, $t \geq 0$, $y \in \mathbb{R}$, of a given càdlàg process $X_t$, $t \geq 0$, is as the Radon-Nikodym derivative of the occupation measure of $X$ with respect to the Lebesgue measure in $\mathbb{R}$. For every Borel measurable function $g : \mathbb{R} \to [0;+\infty)$ and $t > 0$,

$$\int_0^t g(X_s)\, \mathrm{d}s = \int_{\mathbb{R}} g(y)\, L_t^y \mathrm{d}y.$$

Notice that for any càdlàg process $X_t$, $t \geq 0$, and a continuous, non–decreasing process $\zeta_t$, $t \geq 0$,

$$\int_0^{\cdot} g(X_{t-})\, \mathrm{d}\zeta_t = \int_0^{\cdot} g(X_t)\, \mathrm{d}\zeta_t.$$

Thus, in view of theorems stated in the preceding subsection we see that the natural candidates for the local times $L_t^a$ for a self-similar or a (pure-jump) Lévy process $X$ are the limits (if they exist)

$$C^{-1} c^{1/\beta-1} \mathrm{n}^{y,c}(X,t) \text{ and } (f(c))^{-1}\, \mathrm{n}^{y,c}(X,t)$$

respectively.

The possibility of definition of the Brownian local times as the normalized limits of the numbers of interval crossings was first conjectured by Paul Lévy, see [12, Sect 2.4], and then a more general result for continuous semimartingales was proven by El Karoui [9]. Then, the case of semimartingales with the jump part with locally finite total variation was considered in [17]. However when $X$ is a Lévy process with the infinite variation jump part, the situation is more complex. Potential theory of Lévy processes provides the following iff condition [4, Chapt. V, Theorem 1]): the occupation measure $\mu_t$ on the time interval $[0;t]$, $t > 0$, defined by the following formula

$$\int_0^t g(X_s)\,\mathrm{d}s = \int_{\mathbb{R}} g(y)\,\mu_t(\mathrm{d}y)$$

is absolutely continuous with respect to the Lebesgue measure (and in such a case its density is the local time) iff

$$\int_{-\infty}^{+\infty} \Re\left(\frac{1}{1+\Psi(\theta)}\mathrm{d}\theta\right) < +\infty. \tag{65}$$

Thus, for exmple if $X$ is a symmetric Cauchy process with the characteristic exponent $\Psi(\theta) = \tilde{C}_0|\theta|$ then the condition (65) is not satisfied and the local time does not exist. However, Proposition 21 still holds. But even if the local time exists, the possibility of the definition of the local time of $X$ in terms of $\mathrm{n}^{y,c}(X,[0;\cdot])$ is (as far as we know) yet to be investigated. There are many "pathwise" approaches to the definition of local times (see for example [2], [10]); the most similar result to the presented approach (where instead of interval crossings, the consecutive downcrossings from hittings of $y+c$ and $y$ are counted) seems to be [11, Theorem 1.9].

# 4 CLT for the truncated variation of Lévy processes

We define $\{L_t^{c,+}\}_{t\geq 0}$, $\{L_t^{c,-}\}_{t\geq 0}$ by

$$L_t^{c,+} := \sum_{s\leq t} \Delta X_s 1_{\Delta X_s \geq c}, \quad L_t^{c,-} := -\sum_{s\leq t} \Delta X_s 1_{\Delta X_s \leq -c}.$$

Further, we define $f^+ : \mathbb{R}_+ \to \mathbb{R}_+$, $f^- : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f^+(c) := \int_{\mathbb{R}_+} x 1_{[c,1]}(x)\mu(\mathrm{d}x), \quad f^-(c) := \int_{\mathbb{R}_-} (-x) 1_{[-1,-c]}(x)\mu(\mathrm{d}x).$$

The following lemma serves as a definition of processes $\{L_t^+\}_{t\geq 0}$ and $\{L_t^-\}_{t\geq 0}$.

**Lemma 22.** *Let $X$ be a Lévy process with the exponent* (51). *Then the following limits exist*

$$L_t^{c,+} - f^+(c)t \to L_t^+,$$
$$L_t^{c,-} - f^-(c)t \to L_t^-,$$

*as $c \to 0$. The convergences hold in $L^2$ in the uniform topology on compacts. The processes $L^{c,+}$ and $L^{c,-}$ are independent and consequently are $L^+$ and $L^-$. Moreover,*

$$L_t^+ - L_t^- + at = X_t. \tag{66}$$

The proof follows by standard arguments, see e.g. [4, Section I.1], and thus is omitted. We note that the Lévy exponents $\Psi^+, \Psi^-$ of $L^+$ and $L^-$ respectively are

$$\Psi^+(\theta) = \int_{(0,+\infty)} \left(e^{i\theta x} - 1 - \mathrm{i}\theta x 1_{[-1,1]}(x)\right)\mu(\mathrm{d}x), \tag{67}$$

$$\Psi^-(\theta) = \int_{(-\infty,0)} \left(e^{i\theta(-x)} - 1 - \mathrm{i}\theta(-x) 1_{[-1,1]}(x)\right)\mu(\mathrm{d}x).$$

The main result of this section is now presented:

**Theorem 11.** *Let $X$ be a Lévy process with the exponent* (51)*. Then there exists a function $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that*

$$\mathrm{TV}^c(X, [0, t]) - f(c)t \to L_t^+ + L_t^-, \tag{68}$$

*and*

$$\mathrm{UTV}^c(X, [0, t]) - (f(c) + a)t/2 \to L_t^+, \tag{69}$$
$$\mathrm{DTV}^c(X, [0, t]) - (f(c) - a)t/2 \to L_t^+, \tag{70}$$

*as $c \to 0$. The convergences hold in probability in the uniform topology on compacts. Moreover, there exist constants $C_1, C_2 > 0$ such that for $c$ small enough we have*

$$C_1 h(c) \le f(c) \le C_2 h(c), \tag{71}$$

*where*

$$h(c) := \int_{[c,1]} x\mu(\mathrm{d}x) + c^{-1} \int_{(0,c)} x^2 \mu(\mathrm{d}x).$$

## 4.1 Proof of Theorem 11

We shall prove (68) then (69) and (70) will follow by (9) and (66).

For the sake of brevity we skip an easy proof for the case of bounded variation assuming it is infinite, or equivalently

$$\int_{[-1,1]\setminus\{0\}} x\mu(\mathrm{d}x) = +\infty. \tag{72}$$

Throughout this proof we keep $T > 0$ fixed and $c$ is contained in $(0, 1)$.

### Preliminaries

Let us define a sequence of stopping times $\{\tau_k^c\}_{k\ge 0}$ by $\tau_k^c := 0$ and

$$\tau_{k+1}^c := \inf\left\{s > \tau_k^c : \mathrm{TV}^c(X, [0, s]) - \mathrm{TV}^c([0, \tau_k^c])) \ge 2c\right\}. \tag{73}$$

We observe that $\left\{\tau_{k+1}^c - \tau_k^c\right\}_{k\ge 1}$ is an i.i.d. sequence. It is independent of $\tau_1^c$, which however has a different law. This follows by the property (K) of Proposition 9 and the strong Markov property of $X$. We think of $\tau_k^c$ as renewal times, we keep track of their number in $\{N_t^c\}_{t\ge 0}$:

$$N_t^c := \inf\left\{k \in \mathbb{N} : \tau_k^c \ge t\right\}.$$

We also set the expected time of renewal $t_c := \mathbb{E}(\tau_2^c - \tau_1^c)$. By (72) one gets easily

$$\lim_{c\searrow 0} t_c = 0. \tag{74}$$

We define

$$\tilde{t}_c := \min\{\tilde{t}_{c,1}, \tilde{t}_{c,2}, \tilde{t}_{c,3}\}, \tag{75}$$

where

$$\tilde{t}_{c,1} := (8i_c)^{-1}, \qquad i_c := \mu((-\infty, c] \cup [c, +\infty)), \tag{76}$$
$$\tilde{t}_{c,2} := c(4|d_c|)^{-1}, \qquad d_c := a - \int_{[-1,1]\setminus(-c,c)} x\mu(\mathrm{d}x), \tag{77}$$
$$\tilde{t}_{c,3} := c^2(32v_c)^{-1}, \qquad v_c := \int_{(-c,c)} x^2\mu(\mathrm{d}x). \tag{78}$$

Importantly, $t_c$ and $\tilde{t}_c$ are comparable, namely for some $C_1 > 1$ we have

$$t_c \in (C_1^{-1}\tilde{t}_c, C_1\tilde{t}_c). \tag{79}$$

Moreover, for some $C_2 > 0$

$$\mathrm{Var}(\tau_2^c - \tau_1^c) \leq C_2\tilde{t}_c^2, \quad \mathrm{Var}(\tau_1^c) \leq C_2\tilde{t}_c^2. \tag{80}$$

Finally,

$$\lim_{c\searrow 0} \frac{c^2}{t_c} = 0. \tag{81}$$

We postpone proving the above relations to Section 4.1.1.

Let $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be any function such that $g(c) \searrow 0$ and $x > 0$. We consider

$$p_c := \mathbb{P}\left(\sup_{t\leq T} |N_t^c - t/t_c| > x/g(c)\right).$$

By the definition of $N^c$ we have

$$p_c \leq \mathbb{P}\left(\exists_{t\leq T} \tau^c_{\lfloor t/t_c + x/g(c)\rfloor} \leq t\right) + \mathbb{P}\left(\exists_{t\leq T} \tau^c_{\lceil t/t_c - x/g(c)\rceil} \geq t\right) =: p_c^1 + p_c^2.$$

We introduce $\tilde{\tau}_n^c := \tau_n^c - \mathbb{E}\tau_n^c$ and notice that it is a martingale. Moreover, we have $\mathbb{E}\tau_{\lfloor t/t_c + x/g(c)\rfloor} \geq t_c(t/t_c + x/g(c) - 2)$. Thus, by Doob's maximal inequality we estimate

$$\begin{aligned} p_c^1 &\leq \mathbb{P}\left(\exists_{t\leq T} \tilde{\tau}_{\lfloor t/t_c + x/g(c)\rfloor} \leq -t_c(x/g(c) - 2)\right) \leq \frac{\mathrm{Var}\left(\tilde{\tau}_{\lfloor T/t_c + x/g(c)\rfloor}\right)}{[t_c(x/g(c) - 2)]^2} \\ &\leq \frac{\mathrm{Var}(\tau_1^c) + (T/t_c + x/g(c))\mathrm{Var}(\tau_2^c - \tau_1^c)}{[t_c(x/g(c) - 2)]^2}. \end{aligned}$$

Using (79) and (80) one gets that for small enough $c$ and some $C_1 > 0$ we have

$$p_c^1 \leq 2\frac{\mathrm{Var}(\tau_1^c)}{t_c^2 x^2} g(c)^2 + 2T\frac{\mathrm{Var}(\tau_2^c - \tau_1^c)}{t_c}\frac{g(c)^2}{t_c^2 x^2} + 2\frac{\mathrm{Var}(\tau_2^c - \tau_1^c)}{t_c^2 x^2} g(c) \leq C_1\left(\frac{g(c)^2}{t_c} + g(c)\right).$$

A similar proof shows that for some $C_2$ we have $p_c^2 \leq C_2\left(\frac{g(c)^2}{t_c} + g(c)\right)$. In the sequel we will use two particular instances of this inequality. Namely, using (74) and (81) one checks that when $c \to 0$ we have

$$\mathbb{P}\left(\sup_{t\leq T} |N_t^c - t/t_c| \geq x/t_c\right) \to 0, \quad \mathbb{P}\left(\sup_{t\leq T} |N_t^c - t/t_c| \geq x/c\right) \to 0. \tag{82}$$

### The main proof - convergence (68)

We are about ot make a decomposition, which is crucial for our proof. First let

$$s_c(x) := \begin{cases} 0 & x \in [0, 4c) \\ x - 4c & x \geq 4c \end{cases}, \quad x \in \mathbb{R}_+ \tag{83}$$

and for $k \in \mathbb{N}$ we put

$$dT_k^c := \mathrm{TV}^c\big(X, [0, \tau_{k+1}^c]\big) - \mathrm{TV}^c(X, [0, \tau_k^c]) - s_c(|\Delta X_{\tau_{k+1}^c}|). \tag{84}$$

We also set

$$f(c) := f_1(c) + f_2(c), \quad f_1(c) := \frac{\mathbb{E}dT_1^c}{t_c}, \quad f_2(c) = \int_{(0,1]} s_c(x)\mu(\mathrm{d}x). \tag{85}$$

Finally, we preset the aforementioned decomposition

$$\begin{aligned}\mathrm{TV}^c(X,[0,t]) - f(c)t =& \sum_{k=0}^{N_t^c}(dT_k^c - \mathbb{E}dT_k^c) + (N_t^c\,(\mathbb{E}dT_1^c) - f_1(c)t) \\ &+ \left(\sum_{s\le t} s_c(|\Delta X_s|) - f_2(c)t\right) \\ &+ \left(\mathrm{TV}^c(X,[0;t]) - \mathrm{TV}^c\left(X,[0;\tau^c_{N_t^c}]\right)\right) =: I_t^c + J_t^c + G_t^c + H_t^c.\end{aligned} \tag{86}$$

Notice that any jump of size bigger than $4c$ induces increase of the total variation by at least $2c$ and thus a renewal. Therefore the terms $s_c(|\Delta X_{\tau^c_{k+1}}|)$ in (84) cancel with $s_c(|\Delta X_s|)$ in (86) and the decomposition is valid.

We shall prove that

$$I_t^c \to 0, \quad J_t^c \to 0, \quad G_t^c \to (L_t^+ + L_t^-), \quad H_t^c \to 0, \tag{87}$$

when $c \to 0$ and the convergences hold in probability in the uniform topology. Notice that this implies the statement (68) with $f$ given in (85).

To prove the first two convergences in (87) we need to show first that for any $k \in \mathbb{N}$

$$2c \le dT_k^c \le 6c. \tag{88}$$

We denote $J := |\Delta X_{\tau^c_{k+1}}|$, $r_1 := \mathrm{TV}^c\big(X,[0,\tau^c_{k+1})\big) - \mathrm{TV}^c(X,[0,\tau^c_k])$ and $r_2 := \mathrm{TV}^c\big(X,[0,\tau^c_{k+1}]\big) - \mathrm{TV}^c\big(X,[0,\tau^c_{k+1})\big)$. Clearly, we have

$$dT_k^c = r_1 + r_2 - s_c(J), \tag{89}$$

By the definition of $\tau_c^k$ in (73) and the fact that the truncated variation is càdlàg we infer that

$$0 \le r_1 < 2c, \quad r_1 + r_2 \ge 2c. \tag{90}$$

Next, by (35) we have

$$J - 2c \le r_2 \le J. \tag{91}$$

To obtain the bound from above in (88) we observe that $s_c(J) \ge J - 4c$ and calculate using (89)

$$T_k^c \le r_1 + r_2 - J + 4c \le r_1 + 4c \le 6c,$$

where in the second step we have used (91) and in the third one (90).

To prove the bound from below in (88) we split into cases corresponding to (83):

**$J \in [0, 4c)$:** Then $s_c(J) = 0$ and the bound follows by (89) and (90).

**$J \ge 4c$:** Then $s_c(J) = J - 4c$; utilizing (89) we get

$$T_k^c \ge r_2 - J + 4c \ge 2c,$$

where in the first step we have used (90) and in the second one (91).

Having proved (88) we pass to the convergence of $I^c$ in (87). We observe that $\{dT_k^c\}_{k\ge1}$ is an i.i.d sequence (note that $dT_0^c$ is independent but has a different distribution). Fix $x > 0$, by a union bound and Doob's maximal inequality we get

$$\begin{aligned}\mathbb{P}\left(\sup_{t\le T}|I_t^c| \ge x\right) &\le \mathbb{P}\left(N_T^c \ge 2T/t_c\right) + \mathbb{P}\left(\sup_{n<2T/t_c}\Big|\sum_{k=0}^{n}(dT_k^c - \mathbb{E}dT_k^c)\Big| \ge x\right) \\ &\le \mathbb{P}\left(N_T^c \ge 2T/t_c\right) + x^{-2}\sum_{k=0}^{\lfloor 2T/t_c\rfloor}\mathrm{Var}\left(dT_k^c\right).\end{aligned}$$

Using (88) and later the first convergence in (82) and (81) we conclude

$$\mathbb{P}\left(\sup_{t\le T}|I_t^c|\ge x\right)\le \mathbb{P}\left(N_T^c\ge 2T/t_c\right)+x^{-2}\frac{2T}{t_c}\left(16c^2\right)\to 0. \tag{92}$$

We pass to the convergence of $J^c$ in (87). Fix $x>0$, recalling (88)

$$\mathbb{P}\left(\sup_{t\le T}|J_t^c|\ge x\right)\to 0,$$

follows easily from the second convergence in (82). The convergence of the Lévy process $G^c$ holds due to the results of [25, Theorem 1] (see also the English version in arXiv: `http://arxiv.org/pdf/0908.1074v1.pdf`). It suffices to check that the characteristic exponent of $G^c$ given by

$$\Psi_c(\theta):=\int_{(0,+\infty)}\left(e^{i\theta s_c(x)}-1-i\theta s_c(x)1_{x\le 1}\right)\mu(\mathrm{d}x).$$

converges point-wise to the one of $L^+ + L^-$, i.e. $\Psi^+ + \Psi^-$, see (67). Now it is enough to check that $|e^{i\theta s_c(x)}-1-i\theta s_c(x)1_{x\le 1}|\le C\min(x^2,1)$ for some $C>0$ and apply Lebesgue dominated convergence theorem

Finally, we have trivial bounds $0\le H_t^c\le 2c$, which imply the convergence of $H^c$.

### The main proof - estimate (71)

We now pass to the proof of (71). Recall (85), for some $C_1, C_2>0$ we get

$$f(c)\le C_1\left(c/\tilde{t}_c+\int_{(0,1]}s_c(x)\mu(\mathrm{d}x)\right)\le C_2\left(ci_c+|d_c|+c^{-1}v_c+\int_{(0,1]}s_c(x)\mu(\mathrm{d}x)\right),$$

where in the first step we used (88) and (79) and in the second (75). Using (83) we get that $c1_{x\ge 2c}+s_c(x)\le x1_{x\ge 2c}$ and thus by (76) we obtain $ci_c+\int_{(0,1]}s_c(x)\mu(\mathrm{d}x)\le\int_{[2c,1]}x\mu(\mathrm{d}x)=:l(c)$. Consequently

$$f(c)\le C_2\left(|d_c|+c^{-1}v_c+l(c)\right).$$

Further, one checks that for some $C_3>0$ and small enough $c$ one have $|d_c|\le C_2 l(c)$. Now, the bound from above in (71) follows by easy calculation.

Let us turn to the bound from below. Using the elementary inequality $\max(a,b,c)\ge(a+b+c)/3$ we conclude that for some $C_4>0$ we have

$$f(c)\ge C_4\left(ci_c+c^{-1}v_c\right)+\int_{(0,1]}s_c(x)\mu(\mathrm{d}x).$$

Now one proves the bound from below in (71) using elementary arguments.

#### 4.1.1 The proof of properties (79)-(81)

Before the proof we present a lemma, which states a deconcentration property.

**Lemma 23.** *There exist $\varepsilon, p>0$ such that for any centred Lévy process $X$ without Brownian part and with jumps smaller than 1 and for any $t>0$ such that $Var(X_t)\ge 1$ we have*

$$\mathbb{P}\left(X_t>\varepsilon\right)\ge p,\quad \mathbb{P}\left(X_t<-\varepsilon\right)\ge p.$$

*Proof.* Rescaling we may assume that $\mathrm{Var}(X_t) = 1$. Since the Lévy measure $\mu$ of $X$ is concentrated on $[-1,1]$ we have $\mathrm{Var}(X_t) = t\int_{[-1,1]} x^2\mu(\mathrm{d}x)$ and

$$\mathbb{E}e^{X_t} = \exp\left(t\int_{[-1,1]}(e^x - 1 - x)\mu(\mathrm{d}x)\right) \leq \exp\left(t\int_{[0,1]} x^2\mu(\mathrm{d}x)\right) = e.$$

Consequently, $\mathbb{E}e^{|X_t|} \leq 2e$. Using the fact that the process is centred and Chebyshev's inequality we get

$$\begin{aligned}1 = \mathrm{Var}(X_t) = \mathbb{E}X_t^2 &= 2\int_0^{+\infty} x\mathbb{P}\left(|X_t| \geq x\right)\mathrm{d}x \\ &\leq 2\int_0^{10} x\mathbb{P}\left(|X_t| \geq x\right)\mathrm{d}x + 4\int_{10}^{+\infty} xe^{-x}\mathrm{d}x.\end{aligned}$$

One checks easily that $C_1 := 1/2 - 2\int_{10}^{+\infty} xe^{-x}\mathrm{d}x > 0$. Further

$$\begin{aligned}C_1 \leq &\int_0^{C_1} x\mathbb{P}\left(|X_t| \geq x\right)\mathrm{d}x + \int_{C_1}^{10} x\mathbb{P}\left(|X_t| \geq x\right)\mathrm{d}x \\ \leq &C_1^2/2 + 10\mathbb{P}\left(|X_t| \geq C_1\right).\end{aligned}$$

Clearly $C_1 - C_1^2/2 > 0$ and the claim of the lemma follows. □

We introduce two auxiliary processes $X^c$ and $Y^c$:

$$\begin{aligned}X_t^c &:= X_t - (L_t^{c,+} - L_t^{c,-}), \\ Y_t^c &:= X_t - (X_t^c - \mathbb{E}X_t^c).\end{aligned} \tag{93}$$

We notice that $X^c$ has jumps smaller than $c$ and $Y^c$ bigger than $c$ thus these processes are independent. Moreover, the quantities introduced in equations (76)-(78) are related as follows. We have $\mathbb{E}X_t^c = td_c$ and $\mathrm{Var}(X_t^c) = tv_c$. The process $Y^c$ decomposes as a compound Poisson process with the intensity of jumps $i_c$ and a drift process $d_ct$.

To prove $t_c \geq C_1^{-1}\tilde{t}_c$ in (79) it is enough to establish that $\mathbb{P}\left(\tau_2^c - \tau_1^c \leq \tilde{t}_c\right) \leq C$ for a constant $C \in (0,1)$ independent of $c$. By definition (73), the property (E) of of Proposition 9 and the strong Markov property we get

$$\mathbb{P}\left(\tau_2^c - \tau_1^c \leq \tilde{t}_c\right) \leq \mathbb{P}\left(\mathrm{TV}^c\big(X, [\tau_1^c, \tau_1^c + \tilde{t}_c]\big) \geq c\right) = \mathbb{P}\left(\mathrm{TV}^c\big(X, [0, \tilde{t}_c]\big) \geq c\right).$$

Further, by the definition of the truncated variation for any $a < b$ we have $\{\mathrm{TV}^c(X, [a,b]) > 0\} \subset \{\sup_{x\in[a;b]} |f(x) - f(a)| \geq c/2\}$, thus

$$\begin{aligned}\mathbb{P}\left(\tau_2^c - \tau_1^c \leq \tilde{t}_c\right) &\leq \mathbb{P}\left(\sup_{s\leq\tilde{t}_c} |X_s| \geq c/2\right) \\ &\leq \mathbb{P}\left(\exists_{s\leq\tilde{t}_c} |\Delta X_s| \geq c\right) + \mathbb{P}\left(\sup_{s\leq\tilde{t}_c} |X_s^c| \geq c/2\right).\end{aligned} \tag{94}$$

Recalling that $\tilde{t}_c \leq \tilde{t}_{c,1}$ and (76) we obtain

$$\mathbb{P}\left(\exists_{s\leq\tilde{t}_c} |\Delta X_s| \geq c\right) = 1 - \exp(-\tilde{t}_c i_c) \leq 1 - \exp(-\tilde{t}_{c,1} i_c) = 1 - \exp\left(-1/8\right).$$

Recall (77) and $\tilde{t}_c \leq \tilde{t}_{c,2}$. One establishes that for any $s \leq \tilde{t}_c$

$$|\mathbb{E}X_s^c| \leq s|d_c| \leq \tilde{t}_{c,2}|d_c| = c/4.$$

This observation, Doob's maximal inequality, $\tilde{t}_c \leq \tilde{t}_{c,3}$ and (78) imply

$$\mathbb{P}\left(\sup_{s\leq \tilde{t}_c}|X^c_s| \geq c/2\right) \leq \mathbb{P}\left(\sup_{s\leq \tilde{t}_c}|X^c_s - \mathbb{E}X^c_s| \geq c/4\right) \leq 16\frac{\mathrm{Var}(X^c_{\tilde{t}_c})}{c^2} = 16\frac{\tilde{t}_c v_c}{c^2} \leq 16\frac{\tilde{t}_{c,3} v_c}{c^2} = 1/2. \tag{95}$$

Checking $1 - \exp(-1/8) + 1/2 < 1$ and recalling (94) we conclude $t_c \geq C_1^{-1}\tilde{t}_c$ as aforementioned.

The proof of $t_c \leq C_1 \tilde{t}_c$ in in (79) is harder. First, we observe that $\mathrm{TV}^c(f, [a;b]) \geq \sup_{x\in[a;b]}|f(x) - f(a)| - c$ and thus

$$\mathbb{P}\left(\mathrm{TV}^c(X, [0,t]) \geq 2c\right) \geq \mathbb{P}\left(\sup_{s\leq t}|X_s| \geq 3c\right) =: p(c,t). \tag{96}$$

We will show that there exist $p, C > 0$, independent of $c$, such that

$$p(c, C\tilde{t}_c) > p, \tag{97}$$

Having this estimate we use the property (D) of Proposition 9 to get that

$$\begin{aligned} p_n := \mathbb{P}\left(\tau^c_2 - \tau^c_1 \geq nC\tilde{t}_c\right) &\leq \mathbb{P}\left(\mathrm{TV}^c\big(X, [0, nC\tilde{t}_c]\big) \leq 2c\right) \\ &\leq \mathbb{P}\left(\sum_{i=0}^{n-1}\mathrm{TV}^c\big(X, [iC\tilde{t}_c, (i+1)C\tilde{t}_c]\big) \leq 2c\right) \\ &\leq \mathbb{P}\left(\forall_{i\leq n-1}\mathrm{TV}^c\big(X, [iC\tilde{t}_c, (i+1)C\tilde{t}_c]\big) \leq 2c\right) \\ &\leq \left[\mathbb{P}\left(\mathrm{TV}^c\big(X, [0, C\tilde{t}_c]\big) \leq 2c\right)\right]^n \leq (1-p)^n. \end{aligned}$$

The last but one estimate follows by the fact that the increments of the truncated variation depend on the increments of the underlaying process, which in our case are independent and identically distributed. Now proving $t_c \leq C_1\tilde{t}_c$ and the first estimate of (80) is an easy exercise. The second estimate of (80) is analogous and left to the reader.

To prove (97) we consider three cases corresponding to the definition of $\tilde{t}_c$ in (75).

Assume $\tilde{t}_c = \tilde{t}_{c,3}$. Recall (76) and check that

$$\mathrm{Var}(X^c_{100\tilde{t}_{c,3}}/(3c)) = \tilde{t}_{c,3}\frac{1000}{9c^2}v_c = \frac{1000}{9\cdot 32} \geq 1.$$

By Lemma 23 on each interval of length $1000 \cdot \tilde{t}_{c,3}$ the process $X^c$ has a positive chance of growing/decreasing more than $\varepsilon(3c)$. Now set $\hat{t}_{c,3} := C_3\tilde{t}_{c,3} = C_3\tilde{t}_c$ with $C_3 > 1000/\varepsilon$. Using the independence of increments we get

$$\mathbb{P}\left(X^c_{\hat{t}_{c,3}} - \mathbb{E}X^c_{\hat{t}_{c,3}} \geq 3c\right) \geq p_3, \quad \mathbb{P}\left(X^c_{\hat{t}_{c,3}} - \mathbb{E}X^c_{\hat{t}_{c,3}} \leq -3c\right) \geq p_3,$$

for some $p_3 > 0$. We recall that that $Y^c$ and $X^c$ are independent, thus

$$\begin{aligned} p(c, \hat{t}_{c,3}) &\geq \mathbb{P}\left(|X_{\hat{t}_{c,3}}| \geq 3c\right) \\ &\geq \mathbb{P}\left(Y^c_{\hat{t}_{c,3}} \geq 0, X^c_{\hat{t}_{c,3}} - \mathbb{E}X^c_{\hat{t}_{c,3}} \geq 3c\right) + \mathbb{P}\left(Y^c_{\hat{t}_{c,3}} < 0, X^c_{\hat{t}_{c,3}} - \mathbb{E}X^c_{\hat{t}_{c,3}} \leq -3c\right) \\ &\geq p_3\mathbb{P}\left(Y^c_{\hat{t}_{c,3}} \geq 0\right) + p_3\mathbb{P}\left(Y^c_{\hat{t}_{c,3}} < 0\right) = p_3. \end{aligned} \tag{98}$$

Assume $\tilde{t}_c = \tilde{t}_{c,2}$. Recall (77), fix $C_2 \geq 1$ such that

$$|d_c|C_2\tilde{t}_c - C_2^{3/4}c = |d_c|C_2\tilde{t}_{c,2} - C_2^{3/4}c = c(C_2/4 - C_2^{3/4}) \geq 3c,$$

and $\hat{t}_{c,2} := C_2\tilde{t}_{c,2} = C_2\tilde{t}_c$. With this choice

$$p(c,\hat{t}_{c,2}) \geq \mathbb{P}\left(|X_{\hat{t}_{c,2}}| \geq 3c\right) \geq \mathbb{P}\left(|Y^c_{\hat{t}_{c,2}}| \geq |d_c|\hat{t}_{c,2}, |X^c_{\hat{t}_{c,2}} - \mathbb{E}X^c_{\hat{t}_{c,2}}| \leq C_2^{3/4}c\right)$$
$$= \mathbb{P}\left(|Y^c_{\hat{t}_{c,2}}| \geq |d_c|\hat{t}_{c,2}\right)\mathbb{P}\left(|X^c_{\hat{t}_{c,2}} - \mathbb{E}X^c_{\hat{t}_{c,2}}| \leq C_2^{3/4}c\right).$$

Recall description of $Y^c$ below (93). If no jumps of $Y^c$ has occurred until $\hat{t}_{c,2}$ then $Y^c_{\hat{t}_{c,2}} = d_c\hat{t}_{c,2}$ and thus using $\tilde{t}_{c,2} \leq \tilde{t}_{c,1}$ and recalling (76) we obtain

$$\mathbb{P}\left(|Y^c_{\hat{t}_{c,2}}| \geq |d_c|\hat{t}_{c,2}\right) \geq \exp\left\{-C_2\tilde{t}_{c,2}i_c\right\} = \exp\left\{-C_2\tilde{t}_{c,1}i_c\right\} = \exp\left\{-C_2/8\right\}.$$

The second term can be estimated as follows (recall that $C_2 \geq 1$)

$$\mathbb{P}\left(|X^c_{\hat{t}_{c,2}} - \mathbb{E}X^c_{\hat{t}_{c,2}}| \geq C_2^{3/4}c\right) \leq \frac{\mathrm{Var}(X^c_{\hat{t}_{c,2}})}{C_2^{3/2}c^2} = \frac{C_2\tilde{t}_{c,2}v_c}{C_2^{3/2}c^2} \leq \frac{1}{8},$$

where in the last estimate we used the fact that $\tilde{t}_{c,2} \leq \tilde{t}_{c,3}$ and (78). Collecting the results above we get

$$p(c,\hat{t}_{c,2}) \geq 7\exp\left(-C_2/8\right)/8 =: p_2. \tag{99}$$

Assume $\tilde{t}_c = \tilde{t}_{c,1}$. Let us introduce measure $\rho$ on $\mathbb{R}_+$ by $\rho(A) := \mu(A) \cap \mu(-A)$. When $\rho([6c,+\infty)) \geq \rho([c,+\infty))/2$ then the probability of a jump bigger than $6c$ occurring in the interval $[0,\tilde{t}_{c,1}]$ is bigger than $1-\exp(-\tilde{t}_{c,1}i_c/2) = 1-\exp(-1/16)$. Such a jump induces that the supremum is at least $3c$. Thus

$$p(c,\tilde{t}_c) \geq 1-\exp(-1/16).$$

Otherwise, $\rho([c,4c)) \geq \rho([c,+\infty))/2$. We set $\hat{t}_{c,1} := C_1\tilde{t}_{c,1} = C_1\tilde{t}_c$, with $C_1 > 1$ to be adjusted. Recalling (76) we calculate

$$\mathrm{Var}(X^{6c}_{\hat{t}_{c,1}}) = \hat{t}_{c,1}\int_{(0,6c)} x^2\rho(\mathrm{d}x) \geq c^2\hat{t}_{c,1}\mu([c,6c]) \geq c^2C_1\tilde{t}_{c,1}\mu([c,+\infty))/2 = c^2C_1/16.$$

Further we proceed like in the case $\tilde{t}_c = \tilde{t}_{c,3}$ concluding that there exist $C_1 > 1$ and $p_1 > 0$ such that for $\hat{t}_{c,1} := C_1\tilde{t}_c$ we have

$$p(t,\hat{t}_{c,1}) \geq p_1. \tag{100}$$

Now, we have covered all cases listed in (75). By (96) clearly $p(c,\cdot)$ is increasing, using (98), (99) and (100) we obtain

$$p(c,(C_1\vee C_2\vee C_3)\tilde{t}_c) \geq p(c,\hat{t}_{c,1}\vee\hat{t}_{c,2}\vee\hat{t}_{c,3}) \geq p_1\wedge p_2\wedge p_3 > 0,$$

thus establishing (97).

Finally, we are to prove (81). Using (79) end it is enough to show that

$$c^2\mu([c,+\infty)) \to 0, \quad c\int_{[c,1]} x\mu(\mathrm{d}x) \to 0, \quad \int_{(0,c)} y^2\mu(\mathrm{d}y) \to 0.$$

The last assertion is obvious by the assumption on the Lévy measure. To prove the second assertion we notice that $c\int_c^1 x\mu(\mathrm{d}x) = \int_0^1 g_c(x)\mu(\mathrm{d}x)$ where $g_c(x) = cx1_{x\geq c}$. Now the convergence follows observing that $g_c(x) \leq x^2$ and utilizing Lebesgue's dominated convergence theorem. Finally, we infer the first statement $c^2\mu([c,+\infty)) \leq c\int_{[c,1]} x\mu(\mathrm{d}x) + c^2\mu((1,+\infty)) \to 0$.

## 4.2 CLT for the truncated variation of strictly $\alpha-$stable processes

We will consider the strictly $\alpha$-stable processes with the index $\alpha \in (1;2)$ and $\alpha = 1$ separately.

### 4.2.1 CLT for the truncated variation of strictly $\alpha-$stable processes with the index $\alpha \in (1;2)$

First, let $X_t$, $t \geq 0$, be a strictly $\alpha$-stable process with the characteristic exponent (cf. [16, Definition 1.5])

$$\Psi_X(\theta) = \tilde{C}_0 |\theta|^\alpha \left(1 - i\gamma \tan \frac{\pi\alpha}{2} \operatorname{sgn}\theta\right), \tag{101}$$

where $\alpha \in (1;2)$ is the index, $\tilde{C}_0 > 0$ is the scale parameter and $\gamma \in [-1;1]$ is the skewness parameter (cf. [16, formula (1.9) on p. 11]). By direct calculations (see also [29, Remark on pp. 67-68]) one checks that

$$\Psi_X(\theta) = \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{i\theta x} + i\theta x\right) \Pi(\mathrm{d}x),$$

where

$$\Pi(\mathrm{d}x) := \begin{cases} C_1 x^{-1-\alpha}\mathrm{d}x, & x > 0, \\ C_2(-x)^{-1-\alpha}\mathrm{d}x, & x < 0, \end{cases} \tag{102}$$

and $C_1 \geq 0$, $C_2 \geq 0$ are given by

$$\begin{cases} C_1 = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)\cos((2-\alpha)\pi/2)} \frac{1+\gamma}{2} \tilde{C}_0 \\ C_2 = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)\cos((2-\alpha)\pi/2)} \frac{1-\gamma}{2} \tilde{C}_0. \end{cases}$$

From (102) we see that $X$ may be decomposed as the difference of two independent, spectrally positive (and strictly $\alpha$-stable) Lévy processes $L^1$, $L^2 : X = L^1 - L^2$, with the characterisitc exponents given respectively by

$$\Psi_{L^1}(\theta) = C_1 \int_0^{+\infty} \frac{1 - e^{i\theta x} + i\theta x}{x^{\alpha+1}} \mathrm{d}x \quad \text{and} \quad \Psi_{L^2}(\theta) = C_2 \int_0^{+\infty} \frac{1 - e^{i\theta x} + i\theta x}{x^{\alpha+1}} \mathrm{d}x.$$

Notice that the sum $L := L^1 + L^2$ is also a spectrally positive and strictly $\alpha$-stable Lévy process with the characterisitc exponent given by

$$\Psi_L(\theta) = C_0 \int_0^{+\infty} \frac{1 - e^{i\theta x} + i\theta x}{x^{\alpha+1}} \mathrm{d}x,$$

where

$$C_0 = C_1 + C_2 = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)\cos((2-\alpha)\pi/2)} \tilde{C}_0.$$

By direct calculation one also checks that

$$\Psi_L(\theta) = \tilde{C}_0 |\theta|^\alpha \left(1 - i \tan \frac{\pi\alpha}{2} \operatorname{sgn}\theta\right). \tag{103}$$

Let us decompose the process $X$ into the sum of two processes $Y_t$, $t \geq 0$, and $Z_t$, $t \geq 0$, given by

$$Y_t := \sum_{0<s\leq t} (X_t - X_{t-}) 1_{|X_t - X_{t-1}| \geq 1}, \quad Z_t := X_t - Y_t. \tag{104}$$

In other words, we split $X = Y + Z$, where $Z$ has jumps smaller than 1 and $Y$ has jumps bigger or equal to 1. $Y$ is a compound Poisson process with the Lï¿œvy measure

$$\Pi_Y(\mathrm{d}x) := \begin{cases} C_1 x^{-1-\alpha}\mathrm{d}x, & x \geq 1, \\ C_2(-x)^{-1-\alpha}\mathrm{d}x, & x \leq -1. \end{cases}$$

Now let $T^c_t$, $t \geq 0$, be the process given by

$$T^c_t := \mathrm{TV}^c(X, [0,t]) - c^{1-\alpha} A t. \tag{105}$$

where $A$ is given as the following limit

$$A = \lim_{N \to +\infty} \mathbb{E}\left(\mathrm{TV}^1(X, [0; N+1]) - \mathrm{TV}^1(X, [0; N])\right). \tag{106}$$

To justify the definition of the process $T^c$ we need to prove that the limit (106) exists. The fact that the difference $\mathrm{TV}^1(X, [0; N+1]) - \mathrm{TV}^1(X, [0; N])$ converges almost surely, and hence in distribution, as $N \to +\infty$, follows from the reasoning after (111) in the proof of Theorem 12. To prove that this distribution has finite expectation let us notice that by (30) we have $\mathrm{TV}^1(X, [0; N+1]) - \mathrm{TV}^1(X, [0; N]) \leq \mathrm{TV}^1(X, [N; N+1]) + 1$. Now, by (32) we have

$$\mathrm{TV}^1(X, [N; N+1]) \leq \mathrm{TV}^0(Y, [N; N+1]) + \mathrm{TV}^1(Z, [N; N+1]).$$

Recalling that $\mathrm{TV}^0$ is the same as the total variation, we easily calculate

$$\mathbb{E}\mathrm{TV}^0(Y, [N; N+1]) = (C_1 + C_2) \int_1^{+\infty} x x^{-1-\alpha} \mathrm{d}x = \frac{C_0}{\alpha - 1} < +\infty.$$

The fact that $\mathrm{TV}^1(Z, [N; N+1])$ has finite expectation (and even finite exponential moments) follows from [3, Theorem 3]. Thus, the distribution of $\mathrm{TV}^1(X, [0; N+1]) - \mathrm{TV}^1(X, [0; N])$ is stochastically dominated by the distribution of

$$\mathrm{TV}(Y, [N; N+1]) + \mathrm{TV}^1(Z, [N; N+1]) + 1 =^d \mathrm{TV}(Y, [0; 1]) + \mathrm{TV}^1(Z, [0; 1]) + 1$$

which has finite expectation and the same holds for the limit distribution. Hence the limit (106) exists and is finite.

Now we are ready to state.

**Theorem 12.** *Let $\alpha \in (1; 2)$ and $X_t$, $t \geq 0$, be a strictly $\alpha$-stable process with the characteristic exponent given by formula (101), then*

$$T^c \to L^1 + L^2, \tag{107}$$

*where $L^1$ and $L^2$ are two independent, spectrally positive processes such that $X = L^1 - L^2$ and $L^1 + L^2$ is a strictly $\alpha$-stable, spectrally positive process with the characteristic exponent given by formula (103).*

*Proof.* First we will prove the convergence of the finite-dimensional distributions. By the stable scaling property and (31) we get

$$\{T^c_t\}_{t \geq 0} = \left\{c\left(\mathrm{TV}^1\left(c^{-1}X, [0; t]\right) - A c^{-\alpha} t\right)\right\}_{t \geq 0} =^d \left\{c\left(\mathrm{TV}^1\left(X, [0; c^{-\alpha} t]\right) - A c^{-\alpha} t\right)\right\}_{t \geq 0}.$$

Changing the normalisation we are going to study $T^{(n)}_t$, $t \geq 0$, given by

$$T^{(n)}_t := n^{-1/\alpha}\left(\mathrm{TV}^1(X, [0; nt]) - A n t\right).$$

Let us consider $0 < t_1 < t_2 < \ldots < t_n$ and the random vector $(T^{(n)}_{t_1}, T^{(n)}_{t_2}, \ldots T^{(n)}_{t_n})$. We transform it to $\left(T^{(n)}_{t_1}, T^{(n)}_{t_2} - T^{(n)}_{t_1}, \ldots T^{(n)}_{t_n} - T^{(n)}_{t_{n-1}}\right)$. By (29) and (30) we have

$$\begin{aligned} n^{-1/\alpha}\left(\mathrm{TV}^1(X, [t_i; t_{i-1}]) - A n (t_i - t_{i-1})\right) &\leq T^{(n)}_{t_i} - T^{(n)}_{t_{i-1}} \\ &\leq n^{-1/\alpha}\left(\mathrm{TV}^1(X, [t_i; t_{i-1}]) - A n (t_i - t_{i-1})\right) + n^{-1/\alpha} \quad \text{a.s.} \end{aligned}$$

We notice that $\mathrm{TV}^1(X,[t_i;t_{i-1}])$ and $\mathrm{TV}^1(X,[t_j;t_{j-1}])$ are independent whenever $i \neq j$ (since they depend only on the increments of $X$). This proves that the increments of $T^{(n)}$ become independent as $n \to +\infty$. Thus in order to get the second order convergence in the sense of finite dimensional distributions

$$T^c \to^d L, \tag{108}$$

where the limit $L$ is an $\alpha$-stable process with the characteristic exponent (103) one needs to show that any $T^{(n)}_{t_i} - T^{(n)}_{t_{i-1}}$ converges in law to the respective increment of the process $L$. By standard arguments it is enough to prove that $T^{(n)}_1 \to^d L_1$.

By stochastic continuity of the Lévy processes, without loss of generality we may assume that $n$ are natural numbers. We are going to use the version of CLT for stationary sequences. To this end we need a preparatory step in order to "stationarize" the sequence. First we extend the process $X$ backwards in the following way. Let $\tilde{X}$ be independent Lévy process with the same distribution as $X$, then we extend the definition of $X$ for negative times

$$X_t := -\tilde{X}_{-t} \text{ for } t < 0. \tag{109}$$

Using (29) and (30) we deduce that for any $l > 0$ the following relation holds

$$\begin{aligned} n^{-1/\alpha}\left(\mathrm{TV}^1(X,[-l;n]) - \mathrm{TV}^1(X,[-l;0]) - An\right) - n^{1/\alpha} &\leq T^{(n)}_1 \\ &\leq n^{-1/\alpha}\left(\mathrm{TV}^1(X,[-l;n]) - \mathrm{TV}^1(X,[-l;0]) - An\right). \end{aligned} \tag{110}$$

We may write

$$\mathrm{TV}^1(X,[-l;n]) - \mathrm{TV}^1(X,[-l;0]) = \sum_{i=0}^{n-1} \Delta\tilde{T}^l_i,$$

where

$$\Delta\tilde{T}^l_i := \mathrm{TV}^1(X,[-l;i+1]) - \mathrm{TV}^1(X,[-l;i]). \tag{111}$$

Let us now consider $\Delta\tilde{T}^l := \left\{\Delta\tilde{T}^l_i\right\}_{i\geq 0}$ as a random element in $\mathbb{R}^\infty$ space with the product topology. We are going to show that the sequence $\left\{\Delta\tilde{T}^l\right\}_{l\geq 0}$ converges almost surely to some $\Delta\tilde{T} = \left\{\Delta\tilde{T}_i\right\}_{i\geq 0}$. Indeed, we define

$$\mathcal{A}_l := \{\exists t \in (-l;0) : |X_t - X_{t-}| \geq 1\}.$$

By (36) we have that conditionally on $\mathcal{A}_l$ the sequence $\Delta\tilde{T}^l, \Delta\tilde{T}^{l+1}, \Delta\tilde{T}^{l+2}, \ldots$ is constant. Thus in order to show the asserted a.s. convergence it is enough to show that $\mathbb{P}(\mathcal{A}_l) \to 1$. This is however obvious property of the Lévy process (in fact the convergence is even exponential).

From (111) it is easy to notice that

$$\left\{\Delta\tilde{T}^l_1, \Delta\tilde{T}^l_2, \Delta\tilde{T}^l_3, \ldots\right\} =^d \left\{\Delta\tilde{T}^{l+1}_0, \Delta\tilde{T}^{l+1}_1, \Delta\tilde{T}^{l+1}_2, \ldots\right\}.$$

which proves that $\Delta\tilde{T}$ is stationary.

For $i = 1, 2, \ldots$ denote

$$\Delta T_i = \Delta\tilde{T}_i - A. \tag{112}$$

The convergence of $\Delta\tilde{T}^l$ to $\Delta\tilde{T}$ and (110) imply

$$n^{-1/\alpha}\sum_{i=0}^{n-1}\Delta T_i - n^{-1/\alpha} \leq T^{(n)}_1 \leq n^{-1/\alpha}\sum_{i=0}^{n-1}\Delta T_i. \tag{113}$$

Notice also that by the definition of $\Delta T_i$ and $A$ (formula (106)) we have

$$\mathbb{E}\Delta T_i = 0 \tag{114}$$

for $i = 1, 2, \ldots$. To prove the convergence of $T_1^{(n)}$ it is enough to study $n^{-1/\alpha} \sum_{i=0}^{n-1} \Delta T_i$. We are now to make a decomposition of $\Delta \tilde{T}_i$ which will be instrumental in further argument. Recall the definition (104) of processes $Y$ and $Z$, and define

$$\Delta \tilde{T} Y_i := \mathrm{TV}^0(Y, [i, i+1]), \tag{115}$$

$$\Delta \tilde{T} Z_i := \Delta \tilde{T}_i - \Delta \tilde{T} Y_i. \tag{116}$$

We will now list the properties of this sequences used further:

1. $\left\{\Delta \tilde{T} Y_i\right\}_{i \geq 0}$ is an i.i.d. sequence,
2. $\left\{\Delta \tilde{T} Z_i\right\}_{i \geq 0}$ is a stationary sequence,
3. $\left\{\Delta \tilde{T} Z_i\right\}_{i \geq 0}$ is exponentially $\alpha$-mixing,
4. $|\Delta \tilde{T} Z_i|$ has exponential moments.

We are now going to verify the above properties and draw some conclusions.

Property 1. is obvious from the fact that $\Delta \tilde{T} Y_i$'s depend only on the increments of the process $X$ on the disjoint intervals.

Property 2. may be verified easily by the fact that $\left\{\Delta \tilde{T}_i\right\}_{i \geq 0}$ is stationary and $\left\{\Delta \tilde{T} Y_i\right\}_{i \geq 0}$ is an i.i.d. sequence.

For property 3. we recall the reader the notion of mixing (see e.g. [8, p. 420]). Given two $\sigma$-fields $\mathcal{F}$ and $\mathcal{G}$ we define

$$\alpha(\mathcal{F}, \mathcal{G}) := \sup_{A \in \mathcal{F}, B \in \mathcal{G}} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|.$$

Let us define $\mathcal{F}_l := \sigma(\Delta \tilde{T}_1, \ldots, \Delta \tilde{T}_l)$ and $\mathcal{G}_{l+h} := \sigma(\Delta \tilde{T}_{l+h}, \Delta \tilde{T}_{l+h+1}, \ldots)$. We are going to show that there exists a constant $d \in (0;1)$ such that for $h > 2$

$$\alpha(\mathcal{F}_l, \mathcal{G}_{l+h}) \leq d^{h-2}. \tag{117}$$

Let us take some $\mathcal{A} \in \mathcal{F}_l$ and $\mathcal{B} \in \mathcal{G}_{l+h}$ and define the event

$$\mathcal{C} := \{\text{there exists } t \in (l+1; l+h-1) \text{ such that } |X_t - X_{t-}| \geq 1\}.$$

Now we can estimate

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) \geq \mathbb{P}(\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}).$$

From (36) we have that $\mathcal{A}$ and $\mathcal{B}$ are independent conditionally on $\mathcal{C}$, $\mathbb{P}(\mathcal{A} \cap \mathcal{B}|\mathcal{C}) = \mathbb{P}(\mathcal{A}|\mathcal{C})\mathbb{P}(\cap \mathcal{B}|\mathcal{C})$, moreover $\mathcal{A}$ is independent from $\mathcal{C}$. Hence

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}) = \mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{B} \cap \mathcal{C}) \geq \mathbb{P}(\mathcal{A})(\mathbb{P}(\mathcal{B}) - \mathbb{P}(\mathcal{C}^c)).$$

On the other hand

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) \leq \mathbb{P}(\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}) + \mathbb{P}(\mathcal{C}^c) = \mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{B} \cap \mathcal{C}) + \mathbb{P}(\mathcal{C}^c) \leq \mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{B}) + \mathbb{P}(\mathcal{C}^c).$$

Putting things together we get

$$|\mathbb{P}(\mathcal{A} \cap \mathcal{B}) - \mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{B})| \leq \mathbb{P}(\mathcal{C}^c).$$

We recall that the jumps larger then 1 arrive at a Poissonian fashion with some constant intensity. Thus indeed, for some constant $d < 1$ we have $\mathbb{P}(\mathcal{C}^c) \leq d^{h-2}$.

Now we are going to establish property 4. First we notice that by (30) and the definition of $\Delta\tilde{T}_i$ we have

$$|\mathrm{TV}^1(X,[i,i+1]) - \Delta\tilde{T}_i| \le 1,$$

thus it is enough to establish the exponential integrability of $|D_i|$, where $D_i := \mathrm{TV}^1(X,[i,i+1]) - \mathrm{TV}^0(Y,[i,i+1])$. Firstly, directly by (30) we prove that

$$D_i \le \mathrm{TV}^1(X-Y,[i,i+1]) = \mathrm{TV}^1(Z,[i,i+1]) .$$

We note that by [3, Theorem 3] the right-hand side has exponential moments. Secondly, we notice that

$$\mathrm{TV}^0(Y,[i,j]) = \sum_{t\in(i;j]} |X_t - X_{t-}| 1_{\{|X_t - X_{t-}|\ge 1\}}, \tag{118}$$

i.e. it is a compound Poisson process with the Lï¿œvy measure $(C_1 + C_2)\, x^{-1-\alpha} 1_{|x|\ge 1}$. For given $i = 0, 1, \ldots$ let $i < t_1 < t_2 < \ldots < t_n \le i+1$ be all times $t \in (i; i+1]$ such that $|X_t - X_{t-}| \ge 1$ and let $t_0 = 1$, $t_{n+1} = i+1$. By the subaddivity property (29) and then by (35) we have

$$\begin{aligned}
\mathrm{TV}^1(X,[i,i+1]) &\ge \sum_{j=0}^{n} \mathrm{TV}^1(X,[t_j,t_{j+1}]) \\
&= \sum_{j=0}^{n} \mathrm{TV}^1(X,[t_j,t_{j+1})) + \sum_{j=0}^{n} \left(\mathrm{TV}^1(X,[t_j,t_{j+1}]) - \mathrm{TV}^1(X,[t_j,t_{j+1}))\right) \\
&\ge \sum_{j=0}^{n} \mathrm{TV}^1(X,[t_j,t_{j+1})) + \sum_{j=0}^{n-1} \left(\left|X_{t_{j+1}} - X_{t_{j+1}-}\right| - 1\right) \\
&= \sum_{j=0}^{n} \mathrm{TV}^1(Z,[t_j,t_{j+1})) + \mathrm{TV}^0(Y,[i,i+1]) - \sum_{t\in(i,i+1]} 1_{\{|X_t - X_{t-}|\ge 1\}}.
\end{aligned}$$

From this we get that

$$D_i = \mathrm{TV}^1(X,[i,i+1]) - \mathrm{TV}^0(Y,[i,i+1]) \ge - \sum_{t\in(i,i+1]} 1_{\{|X_t - X_{t-}|\ge 1\}}.$$

The last sum is obviously a Poisson random variable hence it is exponentially integrable. Using the obtained estimates for $D_i$ we obtain property 4.

Now, we will use the proven properties to draw some conclusions. By property 4., for $i = 1, 2, \ldots$ we may define

$$\Delta T Z_i := \Delta\tilde{T} Z_i - \mathbb{E}\Delta\tilde{T} Z_i. \tag{119}$$

By the central limit theorem for mixing sequences (see [8, Chapt. 7, Theorem 7.8], conditions can be verified easily) we obtain that

$$n^{-1/2} \sum_{i=0}^{n-1} \Delta T Z_i \to^d \mathcal{N}(0,\sigma^2),$$

for some $\sigma > 0$ as $n \to +\infty$. Thus, since we consider $\alpha \in (1; 2)$, we get

$$n^{-1/\alpha} \sum_{i=0}^{n-1} \Delta T Z_i \to^d 0. \tag{120}$$

By (112), (116), (115) and (119), for $i = 1, 2, \ldots$, we have

$$\begin{aligned}
\Delta T_i &= \Delta\tilde{T} Z_i + \mathrm{TV}^0(Y,[i,i+1]) - A \\
&= \Delta T Z_i + \mathrm{TV}^0(Y,[i,i+1]) + \mathbb{E}\Delta\tilde{T} Z_i - A.
\end{aligned}$$

Since $\mathbb{E}\Delta T_i = \mathbb{E}\Delta T Z_i = 0$ (recall (114) and (119)), we must have

$$\Delta T_i = \Delta T Z_i + \mathrm{TV}^0(Y, [i, i+1]) - \mathbb{E}\mathrm{TV}^0(Y, [i, i+1]) . \tag{121}$$

Now we are going to calculate the convergence of finite-dimensional distributions of $T_1^{(n)}$. By (113), (121) and (120) it is sufficient to consider $n^{-1/\alpha} \sum_{i=0}^{n-1} \Delta TY_i$, where

$$\Delta TY_i := \mathrm{TV}^0(Y, [i, i+1]) - \mathbb{E}\mathrm{TV}^0(Y, [i, i+1])$$

for $i = 1, 2, \ldots$. By (118) we have that

$$\mathbb{E}e^{i\theta \mathrm{TV}^0(Y,[i,i+1])} = \exp\left(-C_0 \int_1^{+\infty} \frac{1 - e^{i\theta x}}{x^{1+\alpha}} \mathrm{d}x\right).$$

By the standard calculations we obtain that

$$\mathbb{E}\mathrm{TV}^0(Y, [i, i+1]) = C_0 \int_1^{\infty} x^{-\alpha} \mathrm{d}x.$$

We thus get

$$\mathbb{E}e^{i\theta \Delta TY_i} = \exp\left(-C_0 \int_1^{+\infty} \frac{1 - e^{i\theta x} + i\theta x}{x^{1+\alpha}} \mathrm{d}x\right).$$

Further we calculate

$$\begin{aligned}
\mathbb{E}e^{i\theta n^{-1/\alpha} \sum_{i=0}^{n-1} \Delta TY_i} &= \exp\left(-nC_0 \int_1^{\infty} \frac{1 - e^{in^{-1/\alpha}\theta x} + in^{-1/\alpha}\theta x}{x^{1+\alpha}} \mathrm{d}x\right) \\
&= \exp\left(-C_0 \int_{n^{-1/\alpha}}^{+\infty} \frac{1 - e^{i\theta x} + i\theta x}{x^{1+\alpha}} \mathrm{d}x\right) \\
&\to \exp\left(-C_0 \int_0^{+\infty} \frac{1 - e^{i\theta x} + i\theta x}{x^{1+\alpha}} \mathrm{d}x\right).
\end{aligned}$$

This way we conclude that the law of $\Delta TY_0$ is in the domain of attraction of an $\alpha$-stable law, whose characteristic exponent is given by (103) and we have the convergence of finite dimensional distributions of $T^c$ to $L^1 + L^2$ as $c \to 0+$.

Now, to obtain the a.s. convergence of the process $T^c$ in the uniform convergence topology we use Theorem 11. □

**Remark 24.** *Let $X$ be a strictly $\alpha$-stable process with the characteristic exponent given by (101) and $A_1$, $A_2$ be defined as*

$$A_1 = \lim_{N \to +\infty} \mathbb{E}\left(\mathrm{UTV}^1(X, [0; N+1]) - UTV^1(X, [0; N])\right)$$

*and*

$$A_1 = \lim_{N \to +\infty} \mathbb{E}\left(\mathrm{DTV}^1(X, [0; N+1]) - DTV^1(X, [0; N])\right).$$

*In completely analogous way as Theorem 12 one may prove more accurate result, namely that for*

$$TU_t^c := \mathrm{UTV}^c(X, [0, t]) - c^{1-\alpha} A_1 t,$$

$$TD_t^c := \mathrm{DTV}^c(X, [0, t]) - c^{1-\alpha} A_2 t$$

*and $T^c$ defined with formula (105) the following joint convergence holds*

$$(T^c, TU^c, TD^c) \to \left(L, L^1, L^2\right).$$

### 4.2.2 CLT for the truncated variation of strictly $1-$stable processes

Now we are going to consider the family of strictly 1-stable processes. Let $X_t$, $t \geq 0$, be a strictly 1-stable process with the characteristic exponent

$$\Psi_X(\theta) = \tilde{C}_0|\theta| + i\eta\theta, \tag{122}$$

with the scale parameter $\tilde{C}_0 > 0$ and the drift $\eta \in \mathbb{R}$. (The characterisitc exponent of a strictly 1-stable process is necessarily of this form.) By direct calculations we check that

$$\Psi_X(\theta) = \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{i\theta x} + i\theta x 1_{|x|\leq 1}\right) \Pi(\mathrm{d}x) + i\eta\theta, \tag{123}$$

where

$$\Pi(\mathrm{d}x) = C_0|x|^{-2}\mathrm{d}x$$

and $C_0 = \tilde{C}_0/\pi$. From (123) we see that $X$ may be decomposed as the difference of two independent, spectrally positive processes $M^1$ and $M^2 : X = M^1 - M^2$, with the characteristic exponents given respectively by

$$\Psi_{M^1}(\theta) = C_0 \int_0^{+\infty} \frac{1 - e^{i\theta x} + i\theta x 1_{|x|\leq 1}}{x^2} \mathrm{d}x + i\eta\theta/2$$

and

$$\Psi_{M^2}(\theta) = C_0 \int_0^{+\infty} \frac{1 - e^{i\theta x} + i\theta x 1_{|x|\leq 1}}{x^2} \mathrm{d}x - i\eta\theta/2.$$

By direct calculations one may check that

$$\int_0^{+\infty} \frac{1 - e^{i\theta x} + i\theta x 1_{|x|\leq 1}}{x^2} \mathrm{d}x = \frac{\pi}{2}|\theta| + i\theta \log|\theta| - i\left(1 - \mathbb{C}\right)\theta,$$

where $\mathbb{C} = \Gamma'(1) \approx 0.57721$ is the Euler-Mascheroni constant. Hence the sum $M := M^1 + M^2$ is a spectrally positive proces with the characteristic exponent

$$\Psi_M(\theta) = \tilde{C}_0|\theta| \left(1 + i\frac{2}{\pi}\mathrm{sgn}\,(\theta)\log|\theta|\right) - i\frac{2\left(1 - \mathbb{C}\right)}{\pi}\tilde{C}_0\theta. \tag{124}$$

Let us set

$$B = \lim_{N\to+\infty} \mathbb{E}\left(\mathrm{TV}^1(X,[0;N+1]) - \mathrm{TV}^1(X,[0;N]) - \mathrm{TV}^0(Y,[N;N+1])\right),$$

where $Y$ is given by (104). The constant $B$ is well defined, since it is equal to the expectation of the variable $\Delta\tilde{T}Z_1$, defined in the proof of Theorem 12 (see (116) and property 4.).

We define

$$T_t^c := \mathrm{TV}^c(X,[0,t]) - 2C_0\log c^{-1}\cdot t - B\cdot t. \tag{125}$$

We have

**Proposition 25.** *Let $X_t$, $t \geq 0$, be a strictly 1-stable process with the characteristic exponent given by formula (122), then*

$$T^c \to M^1 + M^2, \tag{126}$$

*where $M^1$, $M^2$ are two independent, spectrally positive processes such that $X = M^1 - M^2$ and $M^1 + M^2$ is a 1-stable process with the characteristic exponent given by formula (124).*

*Proof.* Again, by the scaling argument and the properties of the truncated variation, to prove the convergence of finite dimensional distributions it is sufficient to prove that

$$T_1^{(n)} \to^d M_1,$$

where

$$T_1^{(n)} = n^{-1}\mathrm{TV}^1(X,[0;n]) - 2C_0\log n - B.$$

We extend the process $X$ for negative times with formula (109) and define $\Delta\tilde{T}_i$ exactly in the same way as in the proof of Theorem 12. Let us define $\Delta\tilde{T}Y_i$ and $\Delta\tilde{T}Z_i$ by formulas (115) and (116). They have the same properties 1.-4. as before. Reasoning similarly as before we get

$$n^{-1}\sum_{i=0}^{n-1}\Delta\tilde{T}_i - n^{-1} - 2C_0\log n - B \le T_1^{(n)} \le n^{-1}\sum_{i=0}^{n-1}\Delta\tilde{T}_i - 2C_0\log n - B. \tag{127}$$

We have

$$n^{-1}\sum_{i=0}^{n-1}\Delta\tilde{T}Z_i \to \mathbb{E}\Delta\tilde{T}Z_1 = B \quad \text{a.s.} \tag{128}$$

as $n \to +\infty$. Next, we study the characteristic function of $n^{-1}\sum_{i=0}^{n-1}\Delta\tilde{T}Y_i - 2C_0\log n$ for $\theta > 0$

$$\mathbb{E}\exp\left(i\theta\left(n^{-1}\sum_{i=0}^{n-1}\Delta\tilde{T}Y_i - 2C_0\log n\right)\right) = \exp\left(-2nC_0\int_1^{+\infty}\frac{1-e^{in^{-1}\theta x}}{x^2}\mathrm{d}x - 2i\theta C_0\log n\right).$$

Let us consider the inner expression

$$E_n(\theta) := -2nC_0\int_1^{+\infty}\frac{1-e^{in^{-1}\theta x}}{x^2}\mathrm{d}x - 2i\theta C_0\log n.$$

By direct calculations for $\theta > 0$ we obtain

$$\begin{aligned}
E_n(\theta) =& - 2nC_0\int_1^{+\infty}\frac{1-e^{in^{-1}\theta x}}{x^2}\mathrm{d}x - 2i\theta C_0\log n\\
=&2\theta C_0\int_{n^{-1}\theta}^{+\infty}\frac{e^{ix}-1}{x^2}\mathrm{d}x - 2i\theta C_0\log n\\
=&2\theta C_0\left(\int_{n^{-1}\theta}^{+\infty}\frac{e^{ix}-1}{x^2}\mathrm{d}x - i\int_{n^{-1}\theta}^{1}x^{-1}\mathrm{d}x - i\log\left(n^{-1}\theta\right)\right) - 2i\theta C_0\log n\\
=&2\theta C_0\int_{n^{-1}\theta}^{+\infty}\frac{e^{ix}-1-ix1_{x\le 1}}{x^2}\mathrm{d}x - 2i\theta C_0\log(n^{-1}\theta) - 2i\theta C_0\log n\\
=&2\theta C_0\int_{n^{-1}\theta}^{+\infty}\frac{e^{ix}-1-ix1_{x\le 1}}{x^2}\mathrm{d}x - 2i\theta C_0\log\theta\\
\to&2\theta C_0\int_{0}^{+\infty}\frac{e^{ix}-1-ix1_{x\le 1}}{x^2}\mathrm{d}x - 2i\theta C_0\log\theta\\
=&-\theta C_0\left(\pi - 2i\left(1-\mathbb{C}\right)\right) - 2i\theta C_0\log\theta
\end{aligned}$$

as $n \to +\infty$. Similarly, for $\theta < 0$ we get

$$E_n(\theta) \to -|\theta|C_0\left(\pi + 2i\left(1-\mathbb{C}\right)\right) + 2i|\theta|C_0\log|\theta|$$

as $n \to +\infty$. Summarizing these calculations, we obtain

$$E_n(\theta) \to -\tilde{C}_0|\theta|\left(1 + i\frac{2}{\pi}\mathrm{sgn}\theta\log|\theta|\right) + i\frac{2\left(1-\mathbb{C}\right)}{\pi}\tilde{C}_0\theta.$$

Since $\Delta\tilde{T}_i = \Delta\tilde{T}Z_i + \Delta\tilde{T}Y_i$, by (127), (128) and the obtained convergence of the characteristic function we get the assertion.

To obtain the a.s. convergence in the uniform convergence topology we again use Theorem 11. □

**Remark 26.** *Let $X$ be a strictly 1-stable process with the characteristic exponent given by (122) and $B_1$, $B_2$ be defined as*

$$B_1 = \lim_{N\to+\infty} \mathbb{E}\left(\mathrm{UTV}^1(X,[0;N+1]) - \mathrm{UTV}^1(X,[0;N]) - \mathrm{UTV}^0(Y,[N;N+1])\right)$$

*and*

$$B_2 = \lim_{N\to+\infty} \mathbb{E}\left(\mathrm{DTV}^1(X,[0;N+1]) - \mathrm{DTV}^1(X,[0;N]) - \mathrm{DTV}^0(Y,[N;N+1])\right).$$

*In an analogous way as Proposition 25 one may prove more accurate result, namely that for*

$$TU_t^c := \mathrm{UTV}^c(X,[0,t]) - \frac{1}{2}C_0\log c^{-1}\cdot t - B_1 t - \frac{1}{2}\eta t,$$
$$TD_t^c := \mathrm{DTV}^c(X,[0,t]) - \frac{1}{2}C_0\log c^{-1}\cdot t - B_2 t + \frac{1}{2}\eta t$$

*and $T^c$ defined with formula (125) the following joint convergence holds*

$$(T^c, TU^c, TD^c) \to (M, M_1, M_2).$$

# Appendix (proof of Theorem 6)

Let $\psi : [0;+\infty) \to [0;+\infty)$ be the Laplace exponent of the process $X$, i.e. for $\lambda \geq 0$ we have

$$\psi(\lambda) = \eta\lambda + \int_{-\infty}^0 \left(e^{\lambda x} - 1 - \lambda x 1_{\{x\geq -1\}}\lambda x\right)\nu(\mathrm{d}x),$$

where $\nu$ is the Lévy measure. Recall, that by the assumption

$$\nu(\mathrm{d}x) = \frac{L(x)}{(-x)^{\alpha+1}}\mathrm{d}x \tag{129}$$

for some non-negative Borel-measurable function $L : (-\infty;0) \to [0;+\infty)$, slowly varying at 0. Without loss of generality we may further assume that $\int_{-\infty}^{-1}|x|\,\Pi(\mathrm{d}x) < +\infty$. If this condition is not satisfied then we may split the process $X$ into the sum of two processes, $X = Y + \tilde{X}$, where $Y_t$, $t \geq 0$, is compound Poisson process obtained by summing up all jumps of $X$ till time $t$, whose magnitutdes are greater than 1. $Y$ has locally finite total variation while $\tilde{X}_t$ is exponentially integrable for any $t \geq 0$. Applying (33) we see that the same normalisation as for $\tilde{X}$ will work for $X$.

We calculate

$$\psi(\lambda) = \eta\lambda + \int_{-\infty}^0 \left(e^{\lambda x} - 1 - \lambda x 1_{\{x\geq -1\}}\lambda x\right)\frac{L(x)}{(-x)^{\alpha+1}}\mathrm{d}x. \tag{130}$$

The function $\psi$ is strictly convex and such that $\lim_{\lambda\to+\infty}\psi(\lambda) = +\infty$. Let $\Phi(0)$ be the largest root of the equation $\psi(\lambda) = 0$. On $[\Phi(0);+\infty)$ the function $\psi$ is strictly increasing and one may define its right-inverse function $\Phi : [0;+\infty) \to [\Phi(0);+\infty)$. Let us now recall briefly the notion of scale functions of spectrally negative processes, $W^{(q)}$ for $q \geq 0$. For more details on these functions see e.g. [16, Sect. 8.2]. For any $q \geq 0$ there exists a continuous and increasing function $W^{(q)} : [0;+\infty) \to [0;+\infty)$ such that for any $\lambda > 0$,

$$\int_0^\infty e^{-\lambda x}W^{(q)}(x)\,\mathrm{d}x = \frac{1}{\psi(\lambda) - q} \quad \text{for } \lambda > \Phi(q).$$

For $q = 0$ we will denote $W^{(q)} = W$. Following [24, Subsection 3.2], we also define

$$Z_u^{(q-\psi(u))}(x) = 1 + (q - \psi(u))\int_0^x e^{-uz}W^{(q)}(z)\,\mathrm{d}z.$$

Notice that the stopping time $T_U^c X$ corresponds to the time $\hat{\tau}_c$ in [24]. Next, denoting $\overline{X}_t = \sup_{0\le s\le t} X_s$, $\underline{X}_t = \inf_{0\le s\le t} X_s$, we get that the variable $\xi_{U,0}^c$ is simply equal $\overline{X}_{T_U^c X} - \underline{X}_{T_U^c X} - c$.

By [24, Theorem 3] for $x, q, r = 0$, $u, v \ge 0$ and $p = -\psi(u)$ we have,

$$\mathbb{E}\left[\exp\left(u\underline{X}_{T_U^c X} + uc\right); \overline{X}_{T_U^c X} < v\right] = \frac{Z_u^{(p)}(c-v)}{Z_u^{(p)}(c)} - e^{uv}\frac{W(c-v)}{W(c)}.$$

The function $v \mapsto \mathbb{E}\exp\left(u\underline{X}_{T_U^c X} + uc\right) I_{\left\{\overline{X}_{T_U^c X} < v\right\}}$ is non-decreasing and, by the definition of $T_U^c X$, it is constant for $v > c$ (since $\overline{X}_{T_U^c X} \le c$ almost surely). Differentiating with respect to $v$, we get

$$\begin{aligned}\mathbb{E}\left[\exp\left(u\underline{X}_{T_U^c X} + uc\right); \overline{X}_{T_U^c X} \in \mathrm{d}v\right] =& \frac{\psi(u)\, e^{-u(c-v)} W(c-v)}{Z_u^{(p)}(c)}\mathrm{d}v \\ & - ue^{uv}\frac{W(c-v)}{W(c)}\mathrm{d}v - e^{uv}\frac{\mathrm{d}W(c-v)}{W(c)}\end{aligned}$$

for $v \in (0; c)$ and $\mathbb{E}\left[\exp\left(u\underline{X}_{T_U^c X} + uc\right); \overline{X}_{T_U^c X} \in \mathrm{d}v\right] = 0$ for $v > c$. Now we calculate

$$\begin{aligned}\mathbb{E}\left[\exp\left(u\underline{X}_{T_U^c X} + uc - u\overline{X}_{T_U^c X}\right)\right] &= \int_0^c e^{-uv}\mathbb{E}\left[\exp\left(u\underline{X}_{T_U^c X} + uc\right); \overline{X}_{T_U^c X} \in \mathrm{d}v\right] \\ &= \frac{\psi(u)}{Z_u^{(p)}(c)} e^{-uc}\int_0^c W(c-v)\,\mathrm{d}v - \frac{u}{W(c)}\int_0^c W(c-v)\,\mathrm{d}v \\ &\quad - \frac{1}{W(c)}\int_0^c \mathrm{d}W(c-v) \\ &= \frac{\psi(u)}{Z_u^{(p)}(c)} e^{-uc}\int_0^c W(v)\,\mathrm{d}v - u\frac{\int_0^c W(v)\,\mathrm{d}v}{W(c)} + 1. \qquad (131)\end{aligned}$$

Notice that $\mathbb{E}|X_1| < +\infty$, this follows from the assumptions imposed on the Lévy measure $\Pi$. We have

$$\mathbb{E}\left(\overline{X}_{T_U^c X} - \underline{X}_{T_U^c X} - c\right) = - \left[\frac{\partial}{\partial u}\frac{\psi(u)\, e^{-uc}}{Z_u^{(p)}(c)}\right]_{u=0}\int_0^c W(v)\,\mathrm{d}v + \frac{\int_0^c W(v)\,\mathrm{d}v}{W(c)},$$

which is finite. Notice that $\psi(0) = 0$, $\left[Z_u^{(p)}(c)\right]_{u=0} = Z_0^{(-\psi(0))}(c) = 1$. Hence, for sufficiently small $c$s,

$$\begin{aligned}\mathbb{E}\left(\overline{X}_{T_U^c X} - \underline{X}_{T_U^c X} - c\right) &= -\psi'(0)\int_0^c W(v)\,\mathrm{d}v + \frac{\int_0^c W(v)\,\mathrm{d}v}{W(c)} \\ &= \left(-\mathbb{E}X_1 + \frac{1}{W(c)}\right)\int_0^c W(v)\,\mathrm{d}v.\end{aligned}$$

Now let us calculate $\mathbb{E}T_U^c X$. By [24, formula (3.21)] we calculate

$$\mathbb{E}e^{-qT_U^c X} = \frac{1}{1 + q\int_0^c W(v)\,\mathrm{d}v}.$$

Hence $T_U^c X$ has exponential distribution with expectation $\mathbb{E}T_U^c X = \int_0^c W(v)\,\mathrm{d}v$ and we have

$$\begin{aligned}\chi_D(c) &= \frac{\theta_D^c}{\eta_D^c} = \frac{\mathbb{E}T_U^c X}{\mathbb{E}\left(\overline{X}_{T_U^c X} - \underline{X}_{T_U^c X} - c\right)} \\ &= \frac{1}{\frac{1}{W(c)} - \mathbb{E}X_1} = \frac{W(c)}{1 - W(c)\,\mathbb{E}X_1}. \qquad (132)\end{aligned}$$

Now we will apply Karamata's theory of regularly varying functions to obtain the assymptotic behaviour of $W(c)$ as $c \to 0+$. For two functions $f, g : (0; +\infty) \to (0; +\infty)$, the relation $f(x) \sim g(x)$ as $x \to 0+$ ($x \to +\infty$ respectively) will mean that $f(x)/g(x) \to 1$ as $x \to 0+$ ($x \to +\infty$ respectively). Differentiating formula (130) we get

$$\psi''(\lambda) = \int_{-\infty}^{0} e^{\lambda x} \frac{L(x)}{(-x)^{\alpha-1}} \mathrm{d}x.$$

Define a slowly varying function at $+\infty$, $l : [0; +\infty) \to [0; +\infty)$, $l(y) = L(1/(-y))$. Substituting $y = -x$ we get

$$\psi''(\lambda) = \int_{0}^{+\infty} e^{-\lambda y} y^{1-\alpha} l(1/y) \mathrm{d}y.$$

Integrating by parts we have

$$\psi''(\lambda) = \lambda \int_{0}^{+\infty} e^{-\lambda y} \left\{ \int_{0}^{y} z^{1-\alpha} l(1/z) \mathrm{d}z \right\} \mathrm{d}y.$$

It is easy to see that $\int_0^y z^{1-\alpha} l(1/z) \mathrm{d}z \sim y^{2-\alpha} l(1/y) / (2-\alpha)$ as $y \to 0+$. (To see this it is enough to substitute $t = 1/z$ and apply [5, Proposition 1.5.10].) By this observation and [5, Theorem 1.7.1'] we get that

$$\psi''(\lambda) \sim \frac{\Gamma(3-\alpha)}{2-\alpha} \lambda^{\alpha-2} l(\lambda) = \Gamma(2-\alpha) \lambda^{\alpha-2} l(\lambda)$$

as $\lambda \to +\infty$. Applying twice [5, Proposition 1.5.8]

$$\psi(\lambda) \sim \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)} \lambda^{\alpha} l(\lambda),$$

as $\lambda \to +\infty$. So $1/\psi(\lambda) \sim (\alpha(\alpha-1)/\Gamma(2-\alpha)) \lambda^{-\alpha} \tilde{l}(\lambda)$ as $\lambda \to +\infty$, where $\tilde{l}(\lambda) = 1/l(\lambda)$ is slowly varying at $+\infty$. Again applying [5, Theorem 1.7.1'] we have

$$W(c) \sim \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} c^{\alpha-1} \tilde{l}(1/c) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \frac{c^{\alpha-1}}{L(-c)}.$$

as $c \to 0+$. This together with (132) implies $\chi_D(c) \sim \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} c^{\alpha-1}/L(-c)$. Notice also, that since $\theta_U^c = \mathbb{E} T_U^c X = \int_0^c W(v) \mathrm{d}v$, $\theta_U^c \sim \frac{(\alpha-1)}{\Gamma(2-\alpha)} c^{\alpha}/L(-c)$.

To conclude the proof we need to justify for any $u > 0$ the limit

$$\lim_{c \to 0+} \frac{\mathbb{P}\left(\xi_{D,0}^c > u/\chi_D(c)\right)}{\theta_U^c} = 0. \tag{133}$$

To do this first let us choose $K > 0$ and split the process $X$ into the sum $X = \tilde{Y} + \tilde{Z}$, where

$$\tilde{Y}_t = \left( \sum_{s \le t} (X_s - X_{s-}) 1_{X_s - X_{s-} \le -K \cdot c} \right) - \left( \int_{-1}^{-K \cdot c} x \Pi(\mathrm{d}x) \right) t.$$

Since $\overline{X}_{T_U^c X} \le c$, we estimate

$$\begin{aligned} \xi_{D,0}^c &= \overline{X}_{T_U^c X} - \underline{X}_{T_U^c X} - c \le -\underline{X}_{T_U^c X} \\ &\le -\underline{\tilde{Y}}_{T_U^c X} + \left(-\underline{\tilde{Z}}_{T_U^c X}\right). \end{aligned}$$

We have

$$\frac{\mathbb{P}\left(\xi^c_{D,0} > u/\chi_D(c)\right)}{\theta^c_U} \leq \frac{\mathbb{P}\left(-\underline{\tilde{Y}}_{T^c_U X} + \left(-\underline{\tilde{Z}}_{T^c_U X}\right) > u/\chi_D(c)\right)}{\theta^c_U}$$
$$\leq \frac{\mathbb{P}\left(-\underline{\tilde{Y}}_{T^c_U X} > u/(2\chi_D(c))\right)}{\theta^c_U} + \frac{\mathbb{P}\left(\left(-\underline{\tilde{Z}}_{T^c_U X}\right) > u/(2\chi_D(c))\right)}{\theta^c_U}. \tag{134}$$

Now, by the Chebyshev inequality and the fact that $T^c_U X$ is a stopping time, we estimate

$$\begin{aligned}(\theta^c_U)^{-1}\mathbb{P}\left(-\underline{\tilde{Y}}_{T^c_U X} > u/(2\chi_D(c))\right) &\leq \frac{\mathbb{E}\sum_{s\leq T^c_U X}|X_s - X_{s-}|\,1_{(X_s - X_{s-})\leq -K\cdot c}}{\theta^c_U u/(2\chi_D(c))} \\ &= \frac{\mathbb{E}T^c_U X\mathbb{E}\sum_{s\leq 1}|X_s - X_{s-}|\,1_{(X_s - X_{s-})\leq -K\cdot c}}{\theta^c_U u/(2\chi_D(c))} \\ &= \frac{2}{u}\chi_D(c)\left(\int_{-\infty}^{-K\cdot c}|x|\,\Pi(\mathrm{d}x)\right) \\ &\sim \frac{2}{u}\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}\frac{c^{\alpha-1}}{L(-c)}\frac{L(-K\cdot c)}{(K\cdot c)^{\alpha-1}} \\ &= \frac{2}{u}\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}\frac{L(-K\cdot c)}{L(-c)}\frac{1}{K^{\alpha-1}}. \end{aligned} \tag{135}$$

Since $\tilde{Z}_t$ and $\tilde{Z}_t^{\,2} - t\mathbb{E}\tilde{Z}_t^2$ are martingales, by Doob's inequality we have

$$\begin{aligned}(\theta^c_U)^{-1}\mathbb{P}\left(-\underline{\tilde{Z}}_{T^c_U X} > u/(2\chi_D(c))\right) &\leq \frac{\mathbb{E}\left(\underline{\tilde{Z}}_{T^c_U X}\right)^2}{\theta^c_U u^2/(2\chi_D(c))^2} \leq 4\frac{\mathbb{E}\tilde{Z}^2_{T^c_U X}}{\theta^c_U u^2/(2\chi_D(c))^2} \\ &= 4\frac{\mathbb{E}T^c_U X\mathbb{E}\tilde{Z}_1^2}{\theta^c_U u^2/(2\chi_D(c))^2} \\ &= \frac{16}{u^2}(\chi_D(c))^2\int_0^{-K\cdot c}x^2\Pi(\mathrm{d}x) \\ &\sim \frac{16}{u^2}\left(\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}\frac{c^{\alpha-1}}{L(-c)}\right)^2\frac{L(-K\cdot c)(K\cdot c)^{2-\alpha}}{(2-\alpha)} \\ &= \frac{16}{u^2}\left(\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)L(-c)}\right)^2\frac{L(-K\cdot c)K^{2-\alpha}}{(2-\alpha)}c^{\alpha}. \end{aligned} \tag{136}$$

From (134), (135) and (136), we get that

$$\limsup_{c\to 0+}\frac{\mathbb{P}\left(\xi^c_{D,0} > u/\chi_D(c)\right)}{\theta^c_U} \leq \frac{2}{u}\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}\frac{1}{K^{\alpha-1}}.$$

Since $K$ may be arbitrary large, we get (133).

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