Chapter 1: Bonus - malus systems

Examples of bonus malus systems

(a) T. Rolski et al. “Stochastic processes for insurance and finance”, page 279 - German bonus-malus system

(b) R. Kaas et al. ”Modern Actuarial Risk Theory Using R”, page 137 - Dutch bonus-malus system

Remark Hunger for bonus

A policy holder may be inclined not to report all claims, when she/he expects that the claim size is less than the premium to be paid next year, when she/he reports the claims

Each bonus malus system has \(l \in \{1, 2, 3, \ldots\} \) classes, and transition from \(i\)th class to \(j\)th class depends on number of claims reported during policy year.

Define

\[
t_{ij}(k) = \begin{cases} 
1 & \text{if the policy gets transferred from } i\text{th class to } j\text{th class}, \\
0 & \text{otherwise}
\end{cases}
\]

when number of claims reported equals \(k\)

Let \(Y_n\) be number of claims reported by a policy holder during \(n\)th policy year and let

\[
q_k^{(n)} = P(Y_n = k)
\]

Probability of transition from \(i\)th class to \(j\)th class, \(p_{ij}^{(n)}\), during \(n\)th policy year equals

\[
p_{ij}^{(n)} = \sum_{k=0}^{\infty} q_k^{(n)} t_{ij}(k)
\]

We assume, that \(Y_n\) is independent on \(n\) and it has Poisson distribution

\[
q_k^{(n)} = e^{-\lambda} \frac{\lambda^k}{k!}
\]

thus

\[
p_{ij} = p_{ij}(\lambda) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} t_{ij}(k)
\]

Let \(X_n\) be policy holder’s class number at the beginning of \(n\)th policy year

\[
X_{n+1} = \phi(X_n, Y_n)
\]

Let us define \(l \times l\) transition matrix
\[
P = \begin{bmatrix}
    p_{11} & p_{12} & \cdots & p_{1l} \\
p_{21} & p_{22} & \cdots & p_{2l} \\
    \vdots & \vdots & \ddots & \vdots \\
p_{l1} & p_{l2} & \cdots & p_{ll}
\end{bmatrix}
\]

and let \( p^{(n)} = \begin{bmatrix} P(X_n = 1) & P(X_n = 2) & \cdots & P(X_n = l) \end{bmatrix} \)

We have

\[
p^{(n+1)} = p^{(n)}P
\]

thus, iterating the above equality, we obtain

\[
p^{(n+1)} = p^{(1)}P^n
\]

Problem: how to calculate \( P^n \) for large \( n \)?

From linear algebra we know, that matrix \( P \) has \( n \) eigenvalues, which in general may be complex, and may coincide.

Let \( \phi_1, \ldots, \phi_l \) be \textbf{right} (column) eigenvectors of \( P \), let \( \psi_1, \ldots, \psi_l \) be \textbf{left} (row) eigenvectors of \( P \) and let \( \theta_1, \ldots, \theta_l \) be their eigenvalues, then for \( \Phi = (\phi_1^T, \ldots, \phi_l^T) \) and

\[
\Psi = \begin{pmatrix}
    \psi_1 \\
    \vdots \\
    \psi_l
\end{pmatrix}
\]

we have

\[
P\Phi = \Phi \text{diag}(\theta_1, \ldots, \theta_l)
\]

\[
\Psi P = \text{diag}(\theta_1, \ldots, \theta_l)\Psi
\]

If \( \Phi^{-1} \) exists, then \( \Psi = \Phi^{-1} \) and

\[
P = \Phi \text{diag}(\theta_1, \ldots, \theta_l)\Phi^{-1}
\]

Now we have

\[
P = \Phi \text{diag}(\theta_1^n, \ldots, \theta_l^n)\Phi^{-1}
\]

and we obtain \textit{spectral representation} of \( P^n \)

\[
P^n = \sum_{i=1}^l \theta_i^n \phi_i^T \psi_i.
\]

**Remark** If eigenvalues \( \theta_1, \ldots, \theta_l \) are distinct, then \( \psi_i \phi_j^T = \delta_{ij} = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{otherwise}
\end{cases} \)

**Definition** A Markov chain with transition matrix \( P = (p_{ij}) \) is said to be \textit{ergodic} if there exists limit

\[
\pi_i = \lim_{n \to \infty} p_{ij}^{(n)}
\]

I.e. \( \pi_i \) does not depend on \( j \) and moreover
Thus \( \pi = [\pi_1 \pi_2 ... \pi_l] \) is a **stationary distribution** of Markov chain, which means that

\[
\pi = \pi P
\]

and

\[
\pi_i = \sum_{j=1}^{l} \pi_j p_{ji}
\]

**Theorem** A Markov chain with transition matrix \( P = (p_{ij}) \) is ergodic iff \( P \) is regular, which means that for some \( n_0 \) and for all \( i,j \), \( p_{ij}^{(n_0)} > 0 \)

Asymptotic average premium

\[
b = \sum_{i=1}^{l} \pi_i b_i,
\]

where \( b_i \) denotes premium for state (class) \( i \)

We define Loimaranta efficiency

\[
e(\lambda) := \frac{\lambda}{b(\lambda)} \frac{\partial}{\partial \lambda} b(\lambda)
\]

Since \( \pi_i \) is a stationary we get

\[
\pi_i = \pi_i(\lambda) = \sum_{j=1}^{l} \pi_j p_{ji}(\lambda)
\]

where

\[
p_{ij} = p_{ij}(\lambda) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} t_{ij}(k)
\]

Thus

\[
\frac{\partial}{\partial \lambda} \pi_i(\lambda) = \sum_{j=1}^{l} \frac{\partial}{\partial \lambda} \pi_j(\lambda) p_{ji}(\lambda) + \sum_{j=1}^{l} \pi_j(\lambda) \frac{\partial}{\partial \lambda} p_{ji}(\lambda)
\]

and

\[
\frac{\partial}{\partial \lambda} b(\lambda) = \sum_{i=1}^{l} \frac{\partial}{\partial \lambda} \pi_i(\lambda) b_i
\]

We need only to calculate
\[
\frac{\partial}{\partial \lambda} p_\beta(\lambda) = \frac{\partial}{\partial \lambda} \left( e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} t_\beta(k) \right)
\]

and notice that

\[
\sum_{i=1}^{l} \frac{\partial}{\partial \lambda} \pi_i(\lambda) = 0
\]

This way we may calculate Loimaranta efficiency