VOLATILITY - HOW TO DEFINE IT AND HOW TO COMPUTE IT

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Constant volatility in the Black-Scholes model

The classical Black-Scholes model (in short - BS model) for evolution of stock prices S_t assumes constant value of model parameters, i.e constant drift μ and constant **volatility** σ:

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \tag{1}$$

where B_t is a standard Brownian motion.
The equation (1) has unique strong solution

$$S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma B_t},$$

which means that for $\mu - \sigma^2/2 \approx 0$ the logarithmic return over shorter time interval of length d in the BS model has normal distribution with parameters $0, \sigma^2 d$.

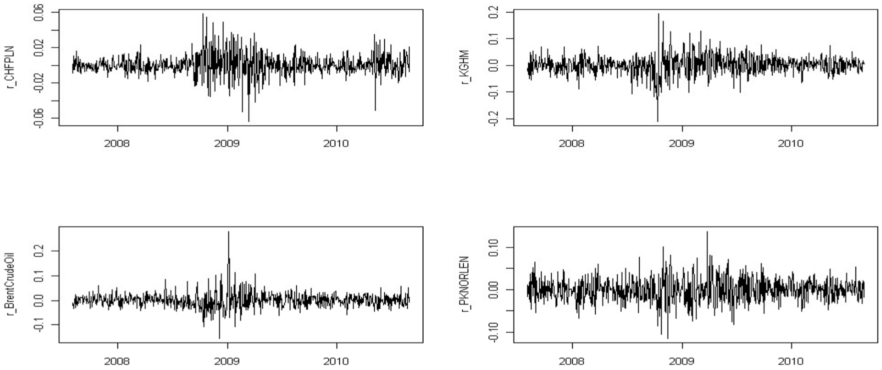
$$\ln\left(S_{s+d}/S_s\right) \sim \mathcal{N}(0, \sigma^2 d).$$

Constant volatility in the Black-Scholes model, cont.

Thus, the parameter σ may be estimated as the proportional to the variance of e.g. daily returns.

Constant volatility in the Black-Scholes model, cont.

Thus, the parameter σ may be estimated as the proportional to the variance of e.g. daily returns. Unfortunately, the assumption about constant volatility in longer time periods is unrealistic.



Actual volatility vs. historical volatility

- The value of the parameter σ which shall be put into BS equation (1) to obtain realistic price process is called **actual volatility**
- The value obtained via the variance estimation of the observed, historical returns is called **historical** or **realized volatility**

The actual volatility is not directly observable, moreover, it may change in time.

Implied volatility

The BS model is often used to price vanilla options, prices of which are observable. The BS equations give closed forms for computing the prices of call and put options, and to do this one needs five parameters, four of which are **observable**:

- current price of the underlying instrument S_0,
- strike price K,
- time to maturity T,
- risk-free interest rate r.

$$Call\_price = C(S_0, K, T, r, \sigma), \quad Put\_price = P(S_0, K, T, r, \sigma).$$

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$$Call\_price = C(S_0, K, T, r, \sigma), \quad Put\_price = P(S_0, K, T, r, \sigma).$$

C and P are strictly increasing functions of σ. Hence, from observable option prices C_{obs}, P_{obs} and four other parameters one may infer the same value of **implied volatility** (put-call parity):

$$\sigma_{impl} : C(S_0, K, T, r, \sigma_{impl}) = C_{obs},$$

$$\sigma_{impl} : P(S_0, K, T, r, \sigma_{impl}) = P_{obs}.$$

Volatility surfaces, smiles and skews

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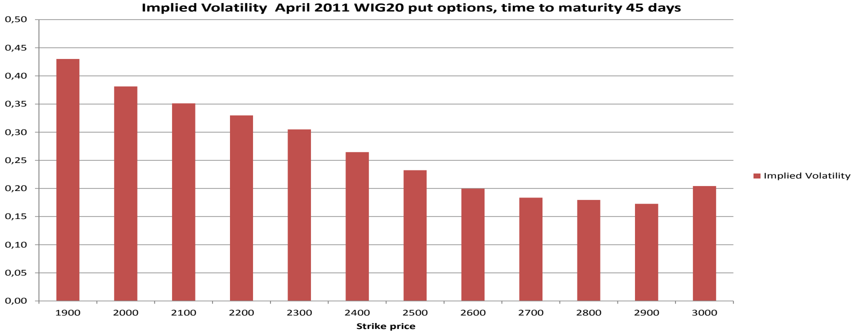
The consequence of the Black-Scholes model shall be flat, horizontal surface of the implied volatility plotted against the strike price K and the time to maturity T.
Plotting implied volatility against the strike price K, one usually obtains valley-shaped (**smile**) or downward slopping curve (**skew**).



Volatility smiles and skews, local volatility models

Though different derivative contracts on the same underlying have different implied volatilities as a function of their own supply and demand dynamics, but they may give us information about market participantss expectation of **future volatility** or they may give us an idea of how desperately they need to hedge.
One can speculate on what he/she thinks implied volatility might be in the future by buying or selling specific portfolios of vanilla options.

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The simplest **local volatility** mathematical models, which incorporate the existence of volatility smiles and skews, assume that actual volatility is a **deterministic** function of time and the current price of the underlying instrument

$$\sigma = \sigma(t, S_t),$$

and the dynamics of the underlying instrument reads as

$$dS_t = \mu S_t dt + \sigma(t, S_t) S_t dB_t.$$

Stochastic volatility models

The next stage in the volatility modeling leads to **stochastic volatility** (SV in short) models, where one treats underlying instrument's volatility as a random process. In a SV framework, volatility changes over time according to a random process usually assigned through a suitable stochastic differential equation (s.d.e.):

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In this manner, SV models manage to reproduce volatility smiles and empirical regularities displayed by stock returns, like sizable negative skewness, fat tails, large kurtosis.

Heston's stochastic volatility model

A widely popular stochastic volatility model, proposed by Heston (1993), assumes that the asset price satisfies the following s.d.e.

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dB_t,$$

where the instantaneous variance $V_t = \sigma_t^2$ follows another s.d.e., namley

$$dV_t = \alpha(\bar{V} - V_t)dt + \beta\sqrt{V_t} dW_t. \tag{2}$$

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In equation (2) $\bar{V} > 0$ corresponds to the long-run variance and $\alpha > 0$ to the mean-reversion force. If additionally $V_0 > 0$ and $2\alpha\bar{V}/\beta^2 \geq 1$ the solution of (2) is always strictly positive (otherwise it remains always non-negative).

Heston's stochastic volatility model, cont.

As largely discussed in the specialized literature, e.g. Das and Sundaram (1999), the effect of stochastic volatility (as measured by the parameter β) is to increase kurtosis of the stock return distribution, whereas a negative correlation parameter ρ generates negative skewness.

However, it has not yet been established whether kurtosis and skewness introduced by stochastic volatility are suitable for the purpose of matching market prices or instead additional refinements of the model are required to achieve this goal.

Heston's stochastic volatility model, cont.

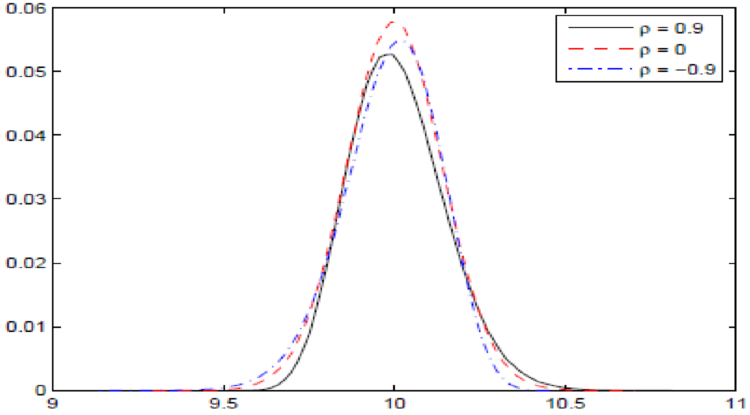

Figure : The effect of ρ on the skewness of the density function. Source: Moodley (2005)

Option's price estimation in Heston's model

For Heston's model it is possible to derive closed analytic form for the characteristic function $f(u) = \int_{-\infty}^{+\infty} e^{iu} p(\ln S_t) d(\ln S_t)$ of the risk-neutral probability density function of the logarithm of S_t.

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To overcome convergence issues Carr and Madan (1999) propose first to calculate the Fourier transform of the modified call price function

$$\tilde{C}(k) = e^{Zk} C(k),$$

where $k = \ln S_T$ and Z is an auxiliary parameter. For $Z > 0$ the function $\tilde{C}$ is square integrable and it may be expressed in terms of the characteristic function $f(u)$.

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Next, the efficient way of inverting the Fourier transform (calculating its inverse Fourier transform) is the application of the Fast Fourier Transform algorithm (FFT).

Heston's model calibration

Another issue is the model calibration. For Heston model the small-time and the large-strike asymptotic formulas are available.

The general conclusion of Bauer (2012) is that it is necessary to take a look at the option data prior the calibration. Then one should choose the appropriate implied volatility asymptotic and reasonable starting values for calibrating the parameters.

Following theses guidelines, one will receive good initial parameters for the calibration with Heston's semi-closed form solution.

Volatility smiles produced by the Heston model

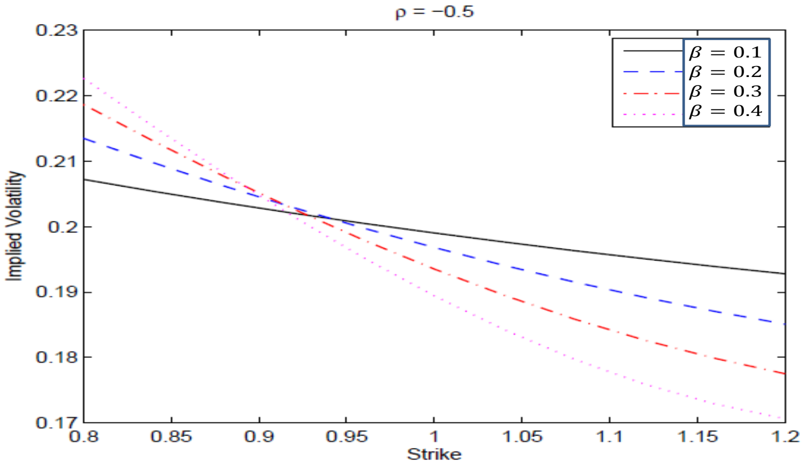

Figure : Volatility smiles produced by the Heston model. Source: Moodley (2005)

What the calibrated model really says us?

- Estimates of the actual volatility based on market implied volatilities outperform, in terms of their predicitve power, straightforward estimates based on historical data.
- Contrary to these findings, subsequent studies of stock index options (cf. Canina and Figlewski (1992)) suggest that the implied volatility has virtually no correlation with the future volatility.
- These results (as well as volatility smiles) may be explained by the presence of **relatively high transaction costs in the case of stock options.**
- **Thus, due to high transaction costs, after calibration we obtain a model which seems to reproduce rather volatility smiles and implied volatilities than actual future volatilities.**

Looking for actual volatility

Looking for some remedy for this, we shall look closer at the definition of the actual volatility itself. In stochastic volatility models there is no direct correspondence between actual volatility and the variance of the returns. (Moreover, the **variance of the returns may be infinite!**)

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However, there exists fundamental correspondence between actual volatility and the quadratic variation of the logarithm of price process:

$$\int_0^T \sigma_t^2 dt =_{Heston} \int_0^T V_t dt = \langle \ln S \rangle_T,$$

where

$$\langle \ln S \rangle_T = \lim_{n \to \infty} \sum_{i=1}^{N} \left(\ln S_{t_i} - \ln S_{t_{i-1}} \right)^2,$$

with $t_i = iT/N$ and the limit is understood as a limit in probability.

Looking for actual volatility, cont.

- So, at least, we shall be able to estimate the integrated realized volatility. However, since we need to use very small time intervals we may encounter here the impact of the market microstructure effects, due to specific trading mechanisms: price formation and discovery, transaction costs, information.

Looking for actual volatility, cont.

- So, at least, we shall be able to estimate the integrated realized volatility. However, since we need to use very small time intervals we may encounter here the impact of the market microstructure effects, due to specific trading mechanisms: price formation and discovery, transaction costs, information.
- To overcome this difficulty some authors (like Barndorff-Nielsen) propose estimators based on multipower variation. E.g. tripower variation given by

$$
TriV\left(\ln S, T\right) := const \sum_{i=0}^{N-3} \left|\ln S_{t_{i+3}} - \ln S_{t_{i+2}}\right|^{2/3} \left|\ln S_{t_{i+2}} - \ln S_{t_{i+1}}\right|^{2/3} \times
$$

$$
\times \left|\ln S_{t_{i+1}} - \ln S_{t_i}\right|^{2/3}.
$$

- These estimators still use very small time intervals.

Another proposal - truncated variation

Alternative proposal for a volatility measure is **truncated variation** at the level $c > 0$, given by

$$TV^c(\ln S, T) := \sup_n \sup_{0 \le t_0 < t_1 < \ldots < t_n \le T} \sum_{i=1}^{n} \max\left\{ \left| \ln S_{t_i} - \ln S_{t_{i-1}} \right| - c, 0 \right\}.$$

Remark

Truncated variation has the following interpretation: it is total variation which takes into account only those jumps, which are larger than c.

Why truncated variation?

- By appropriate choice of the parameter c one may avoid market microstructure effects.
- On the other hand,

$$\lim_{c \downarrow 0} cTV^{c}(\ln S, T) = \langle \ln S \rangle_T \text{ a.s.}$$

- Important from theoretical point of view: it is always finite in opposite to total variation (for any continuous or even càdlàg process) and to calculate it one uses only finite number of summands.

Why truncated variation?, cont.

- Truncated variation may be decomposed into a sum of positive and negative jumps

$$TV^c(\ln S, T) = UTV^c(\ln S, T) + DTV^c(\ln S, T),$$

where

$$UTV^c(\ln S, T) := \sup_n \sup_{0 \le t_0 < t_1 < \ldots < t_n \le T} \sum_{i=1}^{n} \max\left\{\ln S_{t_i} - \ln S_{t_{i-1}} - c, 0\right\},$$

$$DTV^c(\ln S, T) := \sup_n \sup_{0 \le t_0 < t_1 < \ldots < t_n \le T} \sum_{i=1}^{n} \max\left\{\ln S_{t_{i-1}} - \ln S_{t_i} - c, 0\right\}.$$

- Important from practical point of view: UTV and DTV have straightforward interpretation and are easily computable.

Some other interpretation of the upward truncated variation in financial mathematics

Assumptions:

- Dynamics of some financial instrument is described by the stochastic process $S_t > 0$
- There exists proportional transaction costs and $\gamma \in (0, 1)$ is the ratio of transaction value paid as the commission.

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- Dynamics of some financial instrument is described by the stochastic process $S_t > 0$
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Result: maximal possible return from trading this instrument on time interval $[0, T]$ reads as

$$\exp UTV^c (\ln S, T) - 1,$$

where $c = \ln \frac{1+\gamma}{1-\gamma}$.

Sketch of the proof

Let $0 \leq t_{b_1} < t_{s_1} < ... < t_{b_n} < t_{s_n} \leq T$,
t_{b_i} - moments of buying the instrument,
t_{s_j} - moments of selling the instrument.
The price at the moment t reads as $S_t = \exp \ln S_t$ and the return reads as $\prod_{i=1}^{n} \left\{ \frac{S_{t_{s_i}}}{S_{t_{b_i}}} \frac{1-\gamma}{1+\gamma} \right\} - 1.$
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$$\sup_n \sup_{M_n} \prod_{i=1}^{n} \left\{ \frac{S_{t_{s_i}}}{S_{t_{b_i}}} \frac{1-\gamma}{1+\gamma} \right\} = \sup_n \sup_{M_n} \prod_{i=1}^{n} \left\{ \frac{\exp \ln S_{t_{s_i}}}{\exp \ln S_{t_{b_i}}} e^{-c} \right\}$$
$$= \sup_n \sup_{M_n} \exp \left(\sum_{i=1}^{n} \left\{ lnS_{t_{s_i}} - \ln S_{t_{b_i}} - c \right\} \right)$$
$$= \exp UTV^c (\ln S, T).$$

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