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TRUNCATED VARIATION, UPWARD TRUNCATED VARIATION AND DOWNWARD TRUNCATED VARIATION OF BROWNIAN MOTION WITH DRIFT - THEIR MEAN VALUE AND THEIR APPLICATIONS

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Abstract. In [6] for $c > 0$ we defined truncated variation, TV_μ^c, of Brownian motion with drift, $W_t = B_t + \mu t, t \geq 0$, where (B_t) is a standard Brownian motion. In this article we define two related quantities - upward truncated variation

$$UTV_\mu^c[a,b] = \sup_n \sup_{a \leq t_1 < s_1 < \ldots < t_n < s_n \leq b} \sum_{i=1}^n \max\{W_{s_i} - W_{t_i} - c, 0\}$$

and, analogously, downward truncated variation

$$DTV_\mu^c[a,b] = \sup_n \sup_{a \leq t_1 < s_1 < \ldots < t_n < s_n \leq b} \sum_{i=1}^n \max\{W_{t_i} - W_{s_i} - c, 0\}.$$

We prove that exponential moments of the above quantities are finite (in opposite to the regular variation, corresponding to $c = 0$, which is infinite almost surely). We present estimates of the expected value of UTV_μ^c up to universal constants.

As an application we give some estimates of the maximal possible gain from trading a financial asset in the presence of flat commission (proportional to the value of the transaction) when the dynamics of the prices of the asset follows a geometric Brownian motion process. In the presented estimates upward truncated variation appears naturally.

1. Introduction. Let $(B_t, t \geq 0)$ be a standard Brownian motion, and $W_t = B_t + \mu t$ be a Brownian motion with drift μ.

In [6] truncated variation at the level $c > 0$ of Brownian motion with drift μ on the

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interval $[a,b]$ was defined as

$$TV_{\mu}^{c}\left[a,b\right]:=\sup_{n}\sup_{a\leq t_1\leq\cdots\leq t_n\leq b}\sum_{i=1}^{n-1}\max\left\{\left|W_{t_{i+1}}-W_{t_i}\right|-c,0\right\}.$$

(Technical remark: for $a>b$ we set $TV_{\mu}^{c}\left[a,b\right]=0$.)

There were also proved estimates of $\mathbf{E}TV_{\mu}^{c}\left[0,T\right]$ up to universal constants. Using similar techniques as in [6] we will prove existence of finite exponential moments of $TV_{\mu}^{c}\left[0,T\right]$, $\mathbf{E}\exp\left(\alpha TV_{\mu}^{c}\left[0,T\right]\right)$, for any α, $T>0$.

Further we will consider two related quantities

- upward truncated variation, defined as

$$UTV_{\mu}^{c}\left[a,b\right]:=\sup_{n}\sup_{a\leq t_1<s_1<\cdots<t_n<s_n\leq b}\sum_{i=1}^{n}\max\left\{W_{s_i}-W_{t_i}-c,0\right\}$$

- and, analogously, downward truncated variation, defined as

$$DTV_{\mu}^{c}\left[a,b\right]:=\sup_{n}\sup_{a\leq t_1<s_1<\cdots<t_n<s_n\leq b}\sum_{i=1}^{n}\max\left\{W_{t_i}-W_{s_i}-c,0\right\}.$$

It is easy to see that all three above defined quantities have the following properties, which we state only for the truncated variation

- shift invariance property in distributions:

$$\mathcal{L}\left(TV_{\mu}^{c}\left[a,b\right]\right)=\mathcal{L}\left(TV_{\mu}^{c}\left[a+\Delta,b+\Delta\right]\right)$$

- superadditivity property: for any numbers $a\leq a_1<a_2<\cdots<a_n\leq b$

$$TV_{\mu}^{c}\left[a,b\right]\geq\sum_{i=1}^{n-1}TV_{\mu}^{c}\left[a_i,a_{i+1}\right].$$

It is also easy to see that the following relations hold

$$\begin{aligned}
TV_{\mu}^{c}\left[0,T\right] &\geq UTV_{\mu}^{c}\left[0,T\right], &(1)\\
TV_{\mu}^{c}\left[0,T\right] &\geq DTV_{\mu}^{c}\left[0,T\right], &(2)\\
TV_{\mu}^{c}\left[0,T\right] &\leq UTV_{\mu}^{c}\left[0,T\right]+DTV_{\mu}^{c}\left[0,T\right], &\\
\mathcal{L}\left(UTV_{\mu}^{c}\left[0,T\right]\right) &= \mathcal{L}\left(DTV_{-\mu}^{c}\left[0,T\right]\right). &(3)
\end{aligned}$$

By (3) all estimates proved for upward truncated variation have analogs for downward truncated variation.

Analogously as in [6] we will prove some estimates of $\mathbf{E}UTV_{\mu}^{c}\left[0,T\right]$ (and thus for $\mathbf{E}DTV_{\mu}^{c}\left[0,T\right]$) up to universal constants. Unfortunately, the presented estimates involve expected values of some other related variables.

Remark. In order to shorten the proofs we did not put much stress on obtaining the best possible constants in the presented estimates.

Remark. K. Oleszkiewicz pointed out that it would be also interesting to have estimates for higher moments of the defined quantities. However, the author presumes that there are other methods than these used in this paper needed to obtain such estimates.

Remark. A. N. Chuprunov pointed to the author that it would be also interesting to have estimates of quadratic truncated variation, which one may define as

$$QTV_{\mu}^{c}[a,b] := \sup_{n} \sup_{a\leq t_1\leq \dots \leq t_n \leq b} \sum_{i=1}^{n-1} \max\left\{ \left|W_{t_{i+1}} - W_{t_i}\right|^2 - c^2, 0 \right\}.$$

Remark. Similar concept of truncation (or *shrinking*) of random variables on Hilbert spaces investigated Z. J. Jurek in series of his papers beginning with [2], [3], which now evolved in the theory of s-selfdecomposable distributions (see e.g. [4]).

2. Existence of exponential moments of truncated variation. Let us start with the existence of finite exponential moments of $TV_{\mu}^{c}[0,T]$. To prove this let us define

$$T_c = \inf\left\{ t \geq 0 : \sup_{0\leq s\leq t} W_s \geq W_t + c \right\},$$

further let $T_c^{\sup}$ be the last instant when the maximum of W_t on $[0, T_c]$ is attained, and let $T_c^{\inf} \leq T_c^{\sup}$ be such that $W_{T_c^{\inf}} = \inf_{0\leq s\leq T_c^{\sup}} W_s$.

Let us fix $\alpha > 0$ and let $\delta > 0$ be such a small number that

$$1 - \mathbf{E}\exp\left(\alpha \sup_{0\leq t\leq T} W_t + \alpha c\right) P\left(T_c < \delta\right) > 0.$$

By definition of T_c and $T_c^{\inf}$ we have $W_{T_c^{\inf}} > -c$ and $W_{T_c^{\sup}} - W_{T_c^{\inf}} - c \leq W_{T_c^{\sup}}$. Now, by Lemma 1, Lemma 2 in [6] and independence of $W_t - W_{T_c}, t \geq T_c$, and T_c (strong Markov property of Brownian motion) for any $M > 0$ we have

$$\begin{aligned}
\mathbf{E}\exp\left(\alpha TV_{\mu}^{c}[0,T]\wedge M\right) &\leq \mathbf{E}\exp\left(\alpha W_{T_c^{\sup}} + \alpha c + \alpha TV_{\mu}^{c}[T_c,T]\wedge M\right)\\
&\leq \mathbf{E}\exp\left(\alpha W_{T_c^{\sup}} + \alpha c\right)\mathbf{E}\exp\left[\alpha TV_{\mu}^{c}[T_c,T]\wedge M; T_c<\delta\right]\\
&\quad + \mathbf{E}\exp\left(\alpha W_{T_c^{\sup}} + \alpha c\right)\mathbf{E}\exp\left[\alpha TV_{\mu}^{c}[T_c,T]\wedge M; T_c\geq\delta\right]\\
&\leq \mathbf{E}\exp\left(\alpha W_{T_c^{\sup}} + \alpha c\right)\mathbf{E}\exp\left[\alpha TV_{\mu}^{c}[T_c,T+T_c]\wedge M; T_c<\delta\right]\\
&\quad + \mathbf{E}\exp\left(\alpha W_{T_c^{\sup}} + \alpha c\right)\mathbf{E}\exp\left[\alpha TV_{\mu}^{c}[T_c,T+T_c-\delta]\wedge M; T_c\geq\delta\right]\\
&\leq \mathbf{E}\exp\left(\alpha \sup_{0\leq t\leq T} W_t + \alpha c\right)\mathbf{E}\exp\left(\alpha TV_{\mu}^{c}[0,T]\wedge M\right) P\left(T_c<\delta\right)\\
&\quad + \mathbf{E}\exp\left(\alpha \sup_{0\leq t\leq T} W_t + \alpha c\right)\mathbf{E}\exp\left(\alpha TV_{\mu}^{c}[0,T-\delta]\wedge M\right) P\left(T_c\geq\delta\right).
\end{aligned}$$

From the above we have

$$\begin{aligned}
&\mathbf{E}\exp\left(\alpha TV_{\mu}^{c}[0,T]\wedge M\right)\\
&\quad\leq \frac{\mathbf{E}\exp\left(\alpha \sup_{0\leq t\leq T} W_t + \alpha c\right) P\left(T_c\geq\delta\right)}{1-\mathbf{E}\exp\left(\alpha \sup_{0\leq t\leq T} W_t + \alpha c\right) P\left(T_c<\delta\right)}\mathbf{E}\exp\left(\alpha TV_{\mu}^{c}[0,T-\delta]\wedge M\right).
\end{aligned}$$

Similarly

$$\begin{aligned}
&\mathbf{E}\exp\left(\alpha TV_{\mu}^{c}[0,T-\delta]\wedge M\right)\\
&\quad\leq \frac{\mathbf{E}\exp\left(\alpha \sup_{0\leq t\leq T} W_t + \alpha c\right) P\left(T_c\geq\delta\right)}{1-\mathbf{E}\exp\left(\alpha \sup_{0\leq t\leq T} W_t + \alpha c\right) P\left(T_c<\delta\right)}\mathbf{E}\exp\left(\alpha TV_{\mu}^{c}[0,T-2\delta]\wedge M\right).
\end{aligned}$$

Iterating and putting together the above inequalities we finally obtain

$$\mathbf{E}\exp\left(\alpha TV_{\mu}^{c}[0,T]\wedge M\right)\leq\left(\frac{\mathbf{E}\exp\left(\alpha\sup_{0\leq t\leq T}W_t+\alpha c\right)P\left(T_c\geq\delta\right)}{1-\mathbf{E}\exp\left(\alpha\sup_{0\leq t\leq T}W_t+\alpha c\right)P\left(T_c<\delta\right)}\right)^{\lceil T/\delta\rceil}.$$

Letting $M\to\infty$ we get $\mathbf{E}\exp\left(\alpha TV_{\mu}^{c}[0,T]\right)<+\infty$.

By (1) and (2) we obtain the finiteness of exponential moments of $UTV_{\mu}^{c}[0,T]$ and $DTV_{\mu}^{c}[0,T]$ as well.

3. Estimates of expected value of upward and downward truncated variation.

3.1. *Preparatory lemmas.* In order to obtain estimates of $\mathbf{E}UTV_{\mu}^{c}[0,T]$ (and analogously $\mathbf{E}DTV_{\mu}^{c}[0,T]$) we will use similar techniques as in [6]. Due to typographical reasons let us introduce notation $\max\{x,0\}=:(x)_+$.

We will need the following analog of Lemma 2 from [6]:

Lemma 3.1. *We have the following identity*

$$UTV_{\mu}^{c}[0,T]=\sup_{0\leq t<s\leq T_c\wedge T}(W_s-W_t-c)_++UTV_{\mu}^{c}[T_c,T]. \tag{4}$$

Proof. Let $0\leq t_1<s_1<t_2<s_2...<t_n<s_n\leq T$ be numbers from the interval $[0,T]$.

We will prove that

$$\sum_{i=1}^{n}(W_{s_i}-W_{t_i}-c)_+\leq\sup_{0\leq t<s\leq T_c\wedge T}(W_s-W_t-c)_++UTV_{\mu}^{c}[T_c,T]. \tag{5}$$

Let n_0 be the greatest number such that $s_{n_0}<T_c$ and let us assume that $n_0<n$ and $t_{n_0+1}<T_c$.

Let us consider several cases.

- $W_{t_{n_0+1}}\geq W_{T_c}$. In this case
$$\left(W_{s_{n_0+1}}-W_{t_{n_0+1}}-c\right)_+\leq\left(W_{s_{n_0+1}}-W_{T_c}-c\right)_+.$$
and
$$\sum_{i=1}^{n}(W_{s_i}-W_{t_i}-c)_+\leq\sum_{i=1}^{n_0}(W_{s_i}-W_{t_i}-c)_++\left(W_{s_{n_0+1}}-W_{T_c}-c\right)_+ + \sum_{i=n_0+2}^{n}(W_{s_i}-W_{t_i}-c)_+. \tag{6}$$
- $W_{t_{n_0+1}}<W_{T_c}$ and $W_{s_{n_0+1}}\leq W_{T_c^{\sup}}$. In this case $t_{n_0+1}<T_c^{\sup}$ (since for $T_c^{\sup}<t<T_c$, $W_t>W_{T_c}$) so
$$\left(W_{s_{n_0+1}}-W_{t_{n_0+1}}-c\right)_+\leq\left(W_{T_c^{\sup}}-W_{t_{n_0+1}}-c\right)_+$$
and
$$\sum_{i=1}^{n}(W_{s_i}-W_{t_i}-c)_+\leq\sum_{i=1}^{n_0}(W_{s_i}-W_{t_i}-c)_++\left(W_{T_c^{\sup}}-W_{t_{n_0+1}}-c\right)_+ + \sum_{i=n_0+2}^{n}(W_{s_i}-W_{t_i}-c)_+. \tag{7}$$

- $W_{t_{n_0+1}} < W_{T_c}$ and $W_{s_{n_0+1}} > W_{T_c^{\sup}} = W_{T_c} + c$. In this case

$$\begin{aligned}\left(W_{s_{n_0+1}} - W_{t_{n_0+1}} - c\right)_+ &= W_{s_{n_0+1}} - W_{t_{n_0+1}} - c \\ &= W_{T_c^{\sup}} - W_{t_{n_0+1}} - c + W_{s_{n_0+1}} - W_{T_c^{\sup}} \\ &= W_{T_c^{\sup}} - W_{t_{n_0+1}} - c + W_{s_{n_0+1}} - W_{T_c} - c \\ &= \left(W_{T_c^{\sup}} - W_{t_{n_0+1}} - c\right)_+ + \left(W_{s_{n_0+1}} - W_{T_c} - c\right)_+\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{n} (W_{s_i} - W_{t_i} - c)_+ \le \sum_{i=1}^{n_0} (W_{s_i} - W_{t_i} - c)_+ + \left(W_{T_c^{\sup}} - W_{t_{n_0+1}} - c\right)_+ \\ + \left(W_{s_{n_0+1}} - W_{T_c} - c\right)_+ + \sum_{i=n_0+2}^{n} (W_{s_i} - W_{t_i} - c)_+ . \quad (8)\end{aligned}$$

Thus for $t_{n_0+1} < T_c$ inequality (6), (7) or (8) holds and we may assume, adding in the case $t_{n_0+1} < T_c$ new terms in the partition and renaming the old ones, that

$$\begin{aligned}0 &\le t_1 < s_1 < ... < t_{n_0} < s_{n_0} \le T_c, \\ T_c &\le t_{n_0+1} < s_{n_0+1} < ... < t_n < s_n \le T.\end{aligned}$$

In order to prove (5) without loss of generality we may assume that for any $1 \le i \le n_0$, $(W_{s_i} - W_{t_i} - c)_+ > 0$ (otherwise we may omit the summand $(W_{s_i} - W_{t_i} - c)_+$). From definition of T_c we have that for any $1 \le i \le n_0 - 1$, $W_{s_i} - W_{t_{i+1}} < c$, so

$$\begin{aligned}&(W_{s_i} - W_{t_i} - c)_+ + \left(W_{s_{i+1}} - W_{t_{i+1}} - c\right)_+ \\ &= W_{s_i} - W_{t_i} - c + W_{s_{i+1}} - W_{t_{i+1}} - c \\ &= W_{s_{i+1}} - W_{t_i} - c + \left(W_{s_i} - W_{t_{i+1}} - c\right) < W_{s_{i+1}} - W_{t_i} - c.\end{aligned}$$

Iterating the above inequality, we obtain

$$\sum_{i=1}^{n_0} (W_{s_i} - W_{t_i} - c)_+ \le W_{s_{n_0}} - W_{t_1} - c \le \sup_{0 \le t < s \le T_c \wedge T} (W_s - W_t - c)_+ .$$

This, together with the obvious inequality

$$\sum_{i=n_0+1}^{n} (W_{s_i} - W_{t_i} - c)_+ \le \ UTV_\mu^c [T_c, T]$$

proves (5). Taking supremum over all partitions $0 \le t_1 < s_1 < t_2 < s_2 < ... < t_n < s_n \le T$ we finally get

$$UTV_\mu^c [0, T] \le \sup_{0 \le t < s \le T_c \wedge T} (W_s - W_t - c)_+ + UTV_\mu^c [T_c, T] .$$

Since the opposite inequality is obvious, we finally get (4). ∎

Let us now define some auxiliary variables. Let $T_c^{(0)} \equiv 0$ and let $T_c^{(i)}, i = 1, 2, ...$ be defined recursively as

$$T_c^{(i)} = \inf \left\{ t > T_c^{(i-1)} : \sup_{T_c^{(i-1)} \le s \le t} W_s \ge W_t + c \right\} .$$

(notice that $T_c^{(1)} = T_c$). We define a new variable

$$UTV_{\mu}^{c}(T) := \sum_{i=1}^{\infty} e^{-T_c^{(i-1)}/T} \sup_{T_c^{(i-1)} \le t < s \le T_c^{(i)} \wedge \left(T_c^{(i-1)}+T\right)} (W_s - W_t - c)_+ .$$

We have the following

LEMMA 3.2. *The variables $UTV_{\mu}^{c}[0,T]$ and $UTV_{\mu}^{c}(T)$ are related by the following relations*

$$UTV_{\mu}^{c}[0,T] \le eUTV_{\mu}^{c}(T) \tag{9}$$

$$UTV_{\mu}^{c}[0,T] \succeq \frac{1-e^{-1}}{2} UTV_{\mu}^{c}(T) \tag{10}$$

where the first relation holds almost surely and the second holds in the sense of stochastic domination i.e. for every $y \ge 0$, $P\left(UTV_{\mu}^{c}[0,T] \ge y\right) \ge P\left(\frac{1-e^{-1}}{2} UTV_{\mu}^{c}(T) \ge y\right)$.

Proof. By the previous lemma, we have

$$\begin{aligned} UTV_{\mu}^{c}[0,T] &= \sup_{0 \le t < s \le T_c^{(1)} \wedge T} (W_s - W_t - c)_+ + UTV_{\mu}^{c}\left[T_c^{(1)}, T\right] \\ &= \sup_{0 \le t < s \le T_c^{(1)} \wedge T} (W_s - W_t - c)_+ + \sup_{T_c^{(1)} \le t < s \le T_c^{(2)} \wedge T} (W_s - W_t - c)_+ \\ &\quad + UTV_{\mu}^{c}\left[T_c^{(2)}, T\right] \\ &= \ldots = \sum_{i \ge 1 : T_c^{(i-1)} \le T} \sup_{T_c^{(i-1)} \le t < s \le T_c^{(i)} \wedge T} (W_s - W_t - c)_+ . \end{aligned} \tag{11}$$

From (11) we almost immediately get (9)

$$\begin{aligned} UTV_{\mu}^{c}[0,T] &= \sum_{i \ge 1 : T_c^{(i-1)} \le T} \sup_{T_c^{(i-1)} \le t < s \le T_c^{(i)} \wedge T} (W_s - W_t - c)_+ \\ &\le \sum_{i=1}^{\infty} e^{1 - T_c^{(i-1)}/T} \sup_{T_c^{(i-1)} \le t < s \le T_c^{(i)} \wedge \left(T_c^{(i-1)}+T\right)} (W_s - W_t - c)_+ \\ &= eUTV_{\mu}^{c}(T). \end{aligned}$$

In order to prove the second relation let $i_0 \ge 1$ be the greatest index such that $T_c^{(i_0-1)} < T$ and let us consider the term

$$A = \sup_{T_c^{(i_0-1)} \le t < s \le T_c^{(i_0)} \wedge \left(T_c^{(i_0-1)}+T\right)} (W_s - W_t - c)_+ .$$

If $i_0 = 1$ then $A = \sup_{0 \le t < s \le T_c^{(1)} \wedge T} (W_s - W_t - c, 0)_+$, otherwise A is independent from $B = \sup_{0 \le t < s \le T_c^{(1)} \wedge T} (W_s - W_t - c, 0)_+$ but has the same distribution as B.

By (11) we have

$$
\begin{aligned}
UTV_{\mu}^{c}[0,T] &= \sum_{i\geq 1: T_c^{(i-1)}\leq T} \sup_{T_c^{(i-1)}\leq t<s\leq T_c^{(i)}\wedge T} (W_s - W_t - c)_+ \qquad (12)\\
&= \sum_{i=1}^{i_0-1} \sup_{T_c^{(i-1)}\leq t<s\leq T_c^{(i)}} (W_s - W_t - c)_+ \\
&\quad + \sup_{T_c^{(i_0-1)}\leq t<s\leq T} (W_s - W_t - c)_+ .
\end{aligned}
$$

In both cases ($i_0 = 1$ and $i_0 > 1$) $2UTV_{\mu}^{c}[0,T]$ stochastically dominates the sum

$$
S_1 = \sum_{i=1}^{i_0} e^{-T_c^{(i-1)}/T} \sup_{T_c^{(i-1)}\leq t<s\leq T_c^{(i)}\wedge\left(T_c^{(i-1)}+T\right)} (W_s - W_t - c)_+ .
$$

($\sum_{i=1}^{i_0-1} \sup_{T_c^{(i-1)}\leq t<s\leq T_c^{(i)}} (W_s - W_t - c)_+$ dominates the first $i_0 - 1$ terms in the above sum and B, which appears in the sum (12) dominates A.) Similarly, define i_k recursively as the greatest integer such that $T_c^{(i_k-1)} < T_c^{(i_{k-1})} + T$ and

$$
S_k = \sum_{i=i_{k-1}+1}^{i_k} \exp\left(-\frac{T_c^{(i-1)} - T_c^{(i_{k-1})}}{T}\right) \sup_{T_c^{(i-1)}\leq t<s\leq T_c^{(i)}\wedge\left(T_c^{(i-1)}+T\right)} (W_s - W_t - c)_+ .
$$

S_k is independent from $S_1, ..., S_{k-1}$, moreover it has the same distribution as S_1 and

$$
UTV_{\mu}^{c}(T) = \sum_{k=1}^{\infty} e^{-T_c^{(i_{k-1})}/T} S_k.
$$

By definition of $i_k, T_c^{(i_k)} \geq T_c^{(i_{k-1})} + T$, thus we have $T_c^{(i_k)} \geq (k-1)T$. Now, since $2UTV_{\mu}^{c}[0,T] \succeq S_k$, $k = 1, 2, ...,$ we have that

$$
\begin{aligned}
\frac{2}{1-e^{-1}} UTV_{\mu}^{c}[0,T] &= \sum_{k=1}^{\infty} e^{-(k-1)} 2UTV_{\mu}^{c}[0,T] \\
&\succeq \sum_{k=1}^{\infty} e^{-T_c^{(i_{k-1})}/T} 2UTV_{\mu}^{c}[0,T] \\
&\succeq \sum_{k=1}^{\infty} e^{-T_c^{(i_{k-1})}/T} S_k = UTV_{\mu}^{c}(T).
\end{aligned}
$$

which proves (10). ∎

Next, let us state a refinement of Lemma 3 from [6]:

Lemma 3.3. *For any μ and $c > 0$*

$$
P\left(T_c < \frac{1}{3}\mathbf{E}T_c\right) \leq \frac{7}{9}.
$$

Proof. The proof follows exactly as in [6], since one can show that for any real μ

$$
\frac{(\mathbf{E}T_c)^2}{\mathbf{E}T_c^2} = \frac{1}{2}\frac{\left(e^{2\mu c} - 1 - 2\mu c\right)^2}{e^{4\mu c} - 6e^{2\mu c}\mu c + e^{2\mu c} + 2\mu^2 c^2 - 2} \geq \frac{1}{2}
$$

and, by the Paley-Zygmund inequality we obtain

$$P\left(T_c \geq \frac{1}{3}\mathbf{E}T_c\right) \geq \left(1-\frac{1}{3}\right)^2 \frac{(\mathbf{E}T_c)^2}{\mathbf{E}T_c^2} \geq \frac{4}{9}\frac{1}{2} = \frac{2}{9}$$

and

$$P\left(T_c < \frac{1}{3}\mathbf{E}T_c\right) = 1 - P\left(T_c \geq \frac{1}{3}\mathbf{E}T_c\right) \leq \frac{7}{9}.$$

■

3.2. *Estimates for long and short time intervals.* Now we are ready to prove estimates of expected value of $UTV_\mu^c[0,T]$ for long and short time intervals ($T \geq \frac{1}{3}\mathbf{E}T_c$ and $T < \frac{1}{3}\mathbf{E}T_c$ respectively). We have

THEOREM 3.4. *For any $T \geq \frac{1}{3}\mathbf{E}T_c$ we have*

$$\begin{aligned} 0.3\frac{T}{\mathbf{E}T_c}\mathbf{E}\sup_{0\leq t<s\leq T_c\wedge T}(W_s - W_t - c)_+ &\leq \mathbf{E}UTV_\mu^c[0,T] \\ &\leq 27\frac{T}{\mathbf{E}T_c}\mathbf{E}\sup_{0\leq t<s\leq T_c\wedge T}(W_s - W_t - c)_+ . \end{aligned}$$

Proof. By Lemma 3.1 and independence of $W_t - W_{T_c}, t \geq T_c$, and T_c (strong Markov property of Brownian motion) we calculate

$$\begin{aligned} \mathbf{E}UTV_\mu^c[0,T] &= \mathbf{E}\sup_{0\leq t\leq s\leq T_c\wedge T}(W_s - W_t - c)_+ + \mathbf{E}UTV_\mu^c[T_c\wedge T, T] \\ &\leq \mathbf{E}\sup_{0\leq t<s\leq T_c\wedge T}(W_s - W_t - c)_+ + \mathbf{E}\left[UTV_\mu^c[T_c, T]; T_c < \frac{1}{3}\mathbf{E}T_c\right] \\ &\quad + \mathbf{E}\left[UTV_\mu^c[T_c, T]; \frac{1}{3}\mathbf{E}T_c \leq T_c \leq T\right] \\ &\leq \mathbf{E}\sup_{0\leq t<s\leq T_c\wedge T}(W_s - W_t - c)_+ + \mathbf{E}\left[UTV_\mu^c[T_c, T+T_c]; T_c < \frac{1}{3}\mathbf{E}T_c\right] \\ &\quad + \mathbf{E}\left[UTV_\mu^c\left[T_c, T+T_c-\frac{1}{3}\mathbf{E}T_c\right]; \frac{1}{3}\mathbf{E}T_c \leq T_c \leq T\right] \\ &\leq \mathbf{E}\sup_{0\leq t<s\leq T_c\wedge T}(W_s - W_t - c)_+ + \mathbf{E}UTV_\mu^c[0,T]\,P\left(T_c < \frac{1}{3}\mathbf{E}T_c\right) \\ &\quad + \mathbf{E}UTV_\mu^c\left[0, T-\frac{1}{3}\mathbf{E}T_c\right]P\left(T_c \geq \frac{1}{3}\mathbf{E}T_c\right). \end{aligned}$$

Now, by the above inequality and Lemma 3.3

$$\begin{aligned} \mathbf{E}UTV_\mu^c[0,T] &\leq \frac{\mathbf{E}\sup_{0\leq t<s\leq T_c\wedge T}(W_s - W_t - c)_+}{P\left(T_c \geq \frac{1}{3}\mathbf{E}T_c\right)} + \mathbf{E}UTV_\mu^c\left[0, T-\frac{1}{3}\mathbf{E}T_c\right] \\ &\leq \frac{9}{2}\mathbf{E}\sup_{0\leq t<s\leq T_c\wedge T}(W_s - W_t - c)_+ + \mathbf{E}UTV_\mu^c\left[0, T-\frac{1}{3}\mathbf{E}T_c\right]. \end{aligned}$$

Similarly

$$\mathbf{E}UTV_\mu^c\left[0, T-\frac{1}{3}\mathbf{E}T_c\right] \leq \frac{9}{2}\mathbf{E}\sup_{0\leq t<s\leq T_c\wedge T}(W_s - W_t - c)_+ + \mathbf{E}UTV_\mu^c\left[0, T-\frac{2}{3}\mathbf{E}T_c\right].$$

Iterating and putting together the above inequalities we obtain the estimate from above

$$\begin{aligned}
\mathbf{E}UTV_{\mu}^{c}[0,T] &\leq \left\lceil \frac{T}{\frac{1}{3}\mathbf{E}T_c} \right\rceil \frac{9}{2}\mathbf{E} \sup_{0\leq t\leq s\leq T_c\wedge T} (W_s - W_t - c)_+ \\
&\leq \left(\frac{3T}{\mathbf{E}T_c} + 1\right) \frac{9}{2}\mathbf{E} \sup_{0\leq t< s\leq T_c\wedge T} (W_s - W_t - c)_+ \\
&\leq \frac{6T}{\mathbf{E}T_c}\frac{9}{2}\mathbf{E} \sup_{0\leq t\leq s\leq T_c\wedge T} (W_s - W_t - c)_+ \\
&\leq 27\frac{T}{\mathbf{E}T_c}\mathbf{E} \sup_{0\leq t\leq s\leq T_c\wedge T} (W_s - W_t - c)_+ .
\end{aligned}$$

The estimate from below is obtained from Lemma 3.2 (see also the comment after the calculation):

$$\begin{aligned}
\mathbf{E}UTV_{\mu}^{c}[0,T] &\geq \frac{1-e^{-1}}{2}\mathbf{E}UTV_{\mu}^{c}(T) \geq 0.3\mathbf{E}UTV_{\mu}^{c}(T) \\
&= 0.3\sum_{i=1}^{\infty}\mathbf{E}e^{-T_c^{(i-1)}/T} \sup_{T_c^{(i-1)}\leq t< s\leq T_c^{(i)}\wedge\left(T_c^{(i-1)}+T\right)} (W_s - W_t - c)_+ \\
&= 0.3\sum_{i=1}^{\infty}\mathbf{E}e^{-T_c^{(i-1)}/T}\mathbf{E} \sup_{T_c^{(i-1)}\leq t< s\leq T_c^{(i)}\wedge\left(T_c^{(i-1)}+T\right)} (W_s - W_t - c)_+ \\
&= 0.3\left(\sum_{i=1}^{\infty}\left(\mathbf{E}e^{-T_c^{(1)}/T}\right)^{i-1}\right)\mathbf{E} \sup_{0\leq t\leq s\leq T_c\wedge T} (W_s - W_t - c)_+ \\
&= 0.3\frac{1}{1-\mathbf{E}e^{-T_c^{(1)}/T}}\mathbf{E} \sup_{0\leq t< s\leq T_c\wedge T} (W_s - W_t - c)_+ \\
&\geq 0.3\frac{1}{1-\mathbf{E}\left(1-T_c^{(1)}/T\right)}\mathbf{E} \sup_{0\leq t< s\leq T_c\wedge T} (W_s - W_t - c)_+ \\
&= 0.3\frac{T}{\mathbf{E}T_c}\mathbf{E} \sup_{0\leq t\leq s\leq T_c\wedge T} (W_s - W_t - c)_+ .
\end{aligned}$$

In the above calculations we used consecutively: independence of $T_c^{(i-1)}$ and $W_s - W_{T_c^{(i-1)}}$, $s \geq T_c^{(i-1)}$, equality of distributions of every term

$$\sup_{T_c^{(i-1)}\leq t< s\leq T_c^{(i)}\wedge\left(T_c^{(i-1)}+T\right)} (W_s - W_t - c)_+$$

for $i = 1, 2, ...$, definition of $T_c^{(i-1)}$, which implies the equality

$$\mathbf{E}e^{-T_c^{(i-1)}/T} = \left(\mathbf{E}e^{-T_c^{(1)}/T}\right)^{i-1}$$

and finally we used the inequality $e^x \geq 1 + x$. ∎

The estimates in Theorem 3.4 involve expected value of the variable

$$\sup_{0\leq t< s\leq T_c\wedge T} (W_s - W_t - c)_+$$

distribution of which, as far as author knows, is not known, but it may be simulated numerically. We also have

COROLLARY 3.5. *For any $T \geq \frac{1}{3}\mathbf{E}T_c$ we have*

$$\begin{aligned} 3\frac{T}{\mathbf{E}T_c}\mathbf{E}\sup_{0\leq t\leq s\leq \frac{1}{3}\mathbf{E}T_c}(W_s - W_t - c)_+ &\leq \mathbf{E}UTV_\mu^c[0,T] \\ &\leq 27\frac{T}{\mathbf{E}T_c}\mathbf{E}\sup_{0\leq t\leq s\leq T_c}(W_s - W_t - c)_+ . \quad (13) \end{aligned}$$

Proof. The estimate from above is a straightforward consequence of Theorem 3.4 and the estimate from below is obtained immediately by the superadditivity property

$$\begin{aligned} \mathbf{E}UTV_\mu^c[0,T] &\geq \sum_{i=1}^{\lfloor 3T/\mathbf{E}T_c\rfloor}\mathbf{E}UTV_\mu^c\left[\frac{i-1}{3}\mathbf{E}T_c, \frac{i}{3}\mathbf{E}T_c\right] \\ &\geq \lfloor 3T/\mathbf{E}T_c\rfloor\,\mathbf{E}UTV_\mu^c\left[0, \frac{1}{3}\mathbf{E}T_c\right] \\ &\geq 3\frac{T}{\mathbf{E}T_c}\mathbf{E}\sup_{0\leq t\leq s\leq \frac{1}{3}\mathbf{E}T_c}(W_s - W_t - c)_+ . \end{aligned}$$

■

REMARK. Using results of of Hadjiliadis and Vecer [1] we are able to calculate exactly the estimate from above appearing in (13). Using the notation from [1], for $z > 0$ we have

$$\begin{aligned} P\left(\sup_{0\leq t\leq s\leq T_c}(W_s - W_t - c)_+ \geq z\right) &= P\left(\sup_{0\leq t\leq s\leq T_c}(W_s - W_t) \geq z + c\right) \\ &= P\left(T(c, z+c) = T_2(z+c)\right) \end{aligned}$$

and by Theorem 2.1 from [1], for $y > c$ we have

$$P\left(\sup_{0\leq t\leq s\leq T_c}(W_s - W_t) \geq y\right) = \frac{e^{2\mu c} - 2\mu c - 1}{e^{2\mu c} + e^{-2\mu c} - 2}\exp\left(-\frac{2\mu}{e^{2\mu c} - 1}(y - c)\right).$$

Hence

$$\begin{aligned} \mathbf{E}\sup_{0\leq t\leq s\leq T_c}(W_s - W_t - c)_+ &= \int_c^\infty P\left(\sup_{0\leq t\leq s\leq T_c}(W_s - W_t) \geq y\right)dy \\ &= \frac{e^{2\mu c} - 2\mu c - 1}{e^{2\mu c} + e^{-2\mu c} - 2}\int_c^\infty \exp\left(-\frac{2\mu}{e^{2\mu c} - 1}(y - c)\right)dy \\ &= \frac{e^{2\mu c} - 2\mu c - 1}{e^{2\mu c} + e^{-2\mu c} - 2}\,\frac{e^{2\mu c} - 1}{2\mu}. \end{aligned}$$

Estimates of $\mathbf{E}UTV_\mu^c[0,T]$ for short time intervals ($T < \frac{1}{2}\mathbf{E}T_c$) are the subject of the next theorem.

THEOREM 3.6. *For any $T < \frac{1}{3}\mathbf{E}T_c$ we have*

$$\begin{aligned} \mathbf{E}\sup_{0\leq t\leq s\leq T}(W_s - W_t - c)_+ &\leq \mathbf{E}UTV_\mu^c[0,T] \\ &\leq 5\mathbf{E}\sup_{0\leq t\leq s\leq T}(W_s - W_t - c)_+ . \end{aligned}$$

Proof. Applying Lemma 3.1 and independence of $W_t - W_{T_c}, t \geq T_c$, and T_c we again calculate

$$\begin{aligned}
\mathbf{E}UTV_{\mu}^{c}[0,T] &\leq \mathbf{E}\sup_{0\leq t\leq s\leq T_c\wedge T}(W_s - W_t - c)_+ + \mathbf{E}UTV_{\mu}^{c}[T_c\wedge T, T] \\
&\leq \mathbf{E}\sup_{0\leq t\leq s\leq T}(W_s - W_t - c)_+ + \mathbf{E}\left[UTV_{\mu}^{c}[T_c, T]; T_c < T\right] \\
&\leq \mathbf{E}\sup_{0\leq t\leq s\leq T}(W_s - W_t - c)_+ + \mathbf{E}UTV_{\mu}^{c}[0,T]\, P\left(T_c < \frac{1}{3}\mathbf{E}T_c\right) \\
&\leq \mathbf{E}\sup_{0\leq t\leq s\leq T}(W_s - W_t - c)_+ + \mathbf{E}UTV_{\mu}^{c}[0,T]\,\frac{7}{9}.
\end{aligned}$$

Thus we got

$$\mathbf{E}UTV_{\mu}^{c}[0,T] \leq \frac{9}{2}\mathbf{E}\sup_{0\leq t\leq s\leq T}(W_s - W_t - c)_+ .$$

The estimate from above is self-evident

$$\mathbf{E}UTV_{\mu}^{c}[0,T] \geq \mathbf{E}\sup_{0\leq t\leq s\leq T}(W_s - W_t - c)_+ .$$

■

REMARK. In order to calculate the quantity $\mathbf{E}\sup_{0\leq t\leq s\leq T}(W_s - W_t - c)_+$ for $T \leq \frac{1}{3}\mathbf{E}T_c$, which appears in Corollary 3.5 and in Theorem 3.6, one may use results of [5]. Let

$$G_{\bar{D}}(y) = 2e^{\mu y}\left\{L + \sum_{n=1}^{\infty}\frac{\theta_n \sin\theta_n}{\theta_n^2 + \mu^2 y^2 + \mu y}\left(1 - \exp\left(-\frac{\theta_n^2 T}{2y^2} - \frac{\mu^2 T}{2}\right)\right)\right\},$$

where θ_n are positive solutions of the eigenvalue condition $\tan\theta_n = -\frac{\theta_n}{\mu y}$,

$$L = \begin{cases} 0, 0 < y < -\frac{1}{\mu}; \\ \frac{3}{2}\left(1 - e^{-\mu^2 T/2}\right), y = -\frac{1}{\mu}; \\ \frac{2\eta\sinh\eta}{\eta^2 - \mu^2 y^2 - \mu y}\left(1 - \exp\left(\frac{\eta^2 T}{2y^2} - \frac{\mu^2 T}{2}\right)\right), y > -\frac{1}{\mu}; \end{cases}$$

and η is the unique positive solution of $\tanh\eta = -\frac{\eta}{\mu y}$. In the notation used in [5] for $z > 0$ we have

$$\begin{aligned}
P\left(\sup_{0\leq t\leq s\leq T}(W_s - W_t - c)_+ \geq z\right) &= P\left(\sup_{0\leq t\leq s\leq T}(W_s - W_t) \geq z + c\right) \\
&= P\left(\bar{D}(T; -\mu, 1) \geq z + c\right) = G_{\bar{D}}(z + c)
\end{aligned}$$

and thus

$$\mathbf{E}\sup_{0\leq t\leq s\leq T}(W_s - W_t - c)_+ = \int_0^{\infty} G_{\bar{D}}(z + c)\,dz = \int_c^{\infty} G_{\bar{D}}(z)\,dz.$$

However, the above formula is very numerically unstable and it seems not to be a straightforward task to obtain using it good numerical or analytical estimates of expected value of the variable $\sup_{0\leq t\leq s\leq T}(W_s - W_t - c)_+$.

4. Example of application. As it was mentioned earlier, upward truncated variation appears naturally in the expression for **the least upper bound** for the rate of return from any trading of a financial asset, dynamics of which follows geometric Brownian motion, in the presence of flat commission. Similar result was proved in [6] for truncated variation, however, truncated variation is not the least upper bound.

Indeed, similarly as in [6], let us assume that the dynamics of the prices P_t of some financial asset (e.g. stock) is the following $P_t = \exp(\mu t + \sigma B_t)$. We are interested in the maximal possible profit coming from trading this single instrument during time interval $[0, T]$. We buy the instrument at the moments $0 \le t_1 < ... < t_n < T$ and sell it at the moments $s_1 < ... < s_n \le T$, such that $t_1 < s_1 < t_2 < s_2 < ... < t_n < s_n$, in order to obtain the maximal possible profit. Furthermore we assume that for every transaction we have to pay a flat commission and γ is the ratio of the transaction value paid for the commission.

The maximal possible rate of return from our strategy reads as (cf. [6])

$$\sup_n \sup_{0\le t_1<s_1<...<t_n<s_n\le T} \frac{P_{s_1}}{P_{t_1}}\frac{1-\gamma}{1+\gamma}...\frac{P_{s_n}}{P_{t_n}}\frac{1-\gamma}{1+\gamma} - 1.$$

Let M_n be the set of all partitions

$$\pi = \{0 \le t_1 < s_1 < ... < t_n < s_n \le T\}.$$

To see that $\exp\left(\sigma UTV^{c/\sigma}_{\mu/\sigma}[0,T]\right) - 1$ with $c = \ln\frac{1+\gamma}{1-\gamma}$ is the least upper bound for maximal possible rate of return let us substitute

$$\begin{aligned}
\sup_n \sup_{M_n} \prod_{i=1}^n \left\{\frac{P_{s_i}}{P_{t_i}}\frac{1-\gamma}{1+\gamma}\right\} &= \sup_n \sup_{M_n} \prod_{i=1}^n \left\{\frac{\exp(\mu s_i + \sigma B_{s_i})}{\exp(\mu t_i + \sigma B_{t_i})} e^{-c}\right\} \\
&= \sup_n \sup_{M_n} \exp\left(\sigma \sum_{i=1}^n \left\{\left(\frac{\mu}{\sigma}s_i + B_{s_i}\right) - \left(\frac{\mu}{\sigma}t_i + B_{t_i}\right) - \frac{c}{\sigma}\right\}\right) \\
&= \exp\left(\sigma \sup_n \sup_{M_n} \sum_{i=1}^n \left\{\left(\frac{\mu}{\sigma}s_i + B_{s_i}\right) - \left(\frac{\mu}{\sigma}t_i + B_{t_i}\right) - \frac{c}{\sigma}\right\}\right) \\
&= \exp\left(\sigma UTV^{c/\sigma}_{\mu/\sigma}[0,T]\right).
\end{aligned}$$

This gives the claimed bound.

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