On the Young integral, truncated variation and rough paths

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The classical Riemann-Stieltjes integral

We say that the Stieltjes integral

$$\int_a^b f(x) \mathrm{d}g(x)$$

exists in the Riemann sense with the value I, if the sum

$$\sum_{i=1}^{n} f(\xi_i) \{g(x_i) - g(x_{i-1})\},$$

where $a = x_0 < x_1 < \ldots < x_n = b$ and $\xi_i \in [x_{i-1}, x_i]$ is as close to I as we wish, provided the mesh of the partition $\pi = \{x_0, x_1, \ldots, x_n\}$,

$$\mathrm{mesh}(\pi) := \max_{1 \le i \le n} (x_i - x_{i-1}),$$

is sufficiently small.

The classical Riemann-Stieltjes integral cont.

There is no problem with the existence of the Riemann-Stieltjes integral if the **total variation** of the integrator

$$\mathrm{TV}(g,[a,b]) := \sup_n \sup_{a\le x_0<x_1<\ldots<x_n\le b} \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|$$

is finite, the integrand f is bounded and regulated (has left and right limits), and f and g have no common points of discontinuity.

If g is not of bounded variation, then there will be continuous functions which cannot be integrated with respect to g.

By the integration by parts formula

$$\int_a^b f(x)\mathrm{d}g(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(x)\mathrm{d}f(x)$$

it is easy to see that the integral also exists whenever the total variation of f is finite, g is bounded and regulated, and f and g have no common points of discontinuity.

(Strong) p-variation

In his famous paper *An inequality of the Holder type, connected with Stieltjes integration* (Acta Math. 67 (1): 251-282, 1936) Laurence Chisholm Young treated the case where both - the integrand and integrator may have infinite total variations.

He was considering the functions of finite p and q-variations. Let us recall the definition of a (strong) p-variation ($p > 0$): for any $f : [a, b] \to \mathbb{R}$

$$V^p(f, [a, b]) := \sup_n \sup_{a \le x_0 < x_1 < \ldots < x_n \le b} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|^p .$$

This may be generalized and defined for any $f : [a, b] \to E$ attaining its values in a metric space E with the metric d :

$$V^p(f, [a, b]) := \sup_n \sup_{a \le x_0 < x_1 < \ldots < x_n \le b} \sum_{i=1}^{n} d(f(x_i), f(x_{i-1}))^p .$$

(Strong) *p*-variation as a rate-independent measure of irregularity

Strong p-variation may be viewed as a measure of path irregularity.

If $\mathrm{V}^p(f,[a,b]) < +\infty$ for $p \geq 1$ then $\mathrm{V}^q(f,[a,b]) < +\infty$ for all $q > p$.

This folows from inequality $\left(\sum_i |a_i|^q\right)^{1/q} \leq \left(\sum_i |a_i|^p\right)^{1/p}$. This is the reason to introduce the **variation index**:

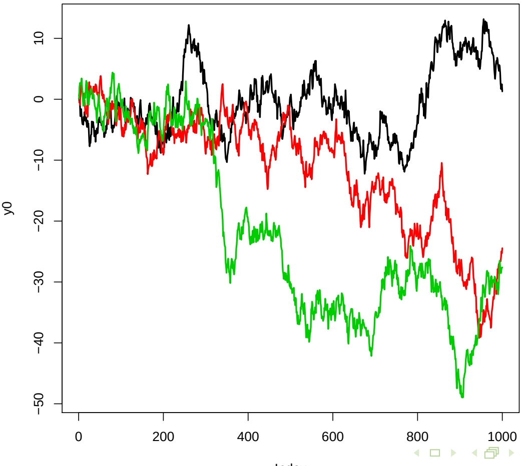
$$Ind_{\mathrm{var}}(f,[a,b]) := \inf\left\{p \geq 1 : \mathrm{V}^p(f,[a,b]) < +\infty\right\}.$$

Another measure of path irregularity is **Hölder exponent**: we say that the function $f : [a,b] \to \mathbb{R}$ is Hölder continuous with the Hölder exponent α if

$$\sup_{a \leq s < t \leq b} \frac{|f(t) - f(s)|}{|t-s|^\alpha} < +\infty.$$

If f is Hölder continuous with the Hölder exponent $0 < \alpha \leq 1$ then $\mathrm{V}^{1/\alpha}(f,[a,b]) < +\infty$.

Examples of paths of a standard Brownian motion, which have almost surely variation index 2 and any Hölder exponent > 0.5



The Loeve-Young inequality

Young (together with Loeve) proved the following estimate: if
$a = x_0 < x_1 < \ldots < x_n = b$, $\xi_i \in [x_{i-1}, x_i]$, and similarly
$a = x'_0 < x'_1 < \ldots < x'_n = b$, $\xi'_i \in \left[x'_{i-1}, x'_i\right]$, and $p > 0$, $q > 0$ then

$$\left|\sum_{i=1}^{n} f\left(\xi_i\right)\left\{g\left(x_i\right) - g\left(x_{i-1}\right)\right\} - \sum_{i=1}^{n} f\left(\xi'_i\right)\left\{g\left(x'_i\right) - g\left(x'_{i-1}\right)\right\}\right|$$
$$\leq 2\left(1 + \zeta\left(\frac{1}{p} + \frac{1}{q}\right)\right)\left(\mathrm{V}^p(f, [a, b])\right)^{1/p}\left(\mathrm{V}^q(g, [a, b])\right)^{1/q}. \qquad (1)$$

Here, ζ denotes the famous Riemann zeta function,

$$\zeta(\alpha) = \sum_{k=1}^{\infty} \frac{1}{k^{\alpha}}.$$

The Young integral

As a result of the Loeve-Young estimate we get

Theorem (L. C. Young, 1936)

The Stieltjes integral $\int_a^b f(x)\mathrm{d}g(x)$ *exists in the Riemann sense whenever* f *and* g *have no common discontinuities, and* $V^p(f,[a,b]) < +\infty$, $V^q(g,[a,b]) < +\infty$ *for some* $p > 0$, $q > 0$ *such that* $p^{-1} + q^{-1} > 1$.

Moreover, for any $\xi \in [a,b]$ *one has the following estimate*

$$\left|\int_a^b f(x)\mathrm{d}g(x) - f(\xi)\left[g(b) - g(a)\right]\right| \leq 2\left(1 + \zeta\left(\frac{1}{p} + \frac{1}{q}\right)\right)\left(V^p(f,[a,b])\right)^{1/p}\left(V^q(g,[a,b])\right)^{1/q}.$$

Remark

Young also provided an example of f *and* g *with finite strong 2-variations and such that* $\int_a^b f(x)\mathrm{d}g(x)$ **does not exists** *in the Riemann sense, and outlined a counter-example for other* p *and* q *such that* $p^{-1} + q^{-1} = 1$.

(Strong) Φ-variation

In his later work *General inequalities for Stieltjes integrals and the convergence of Fourier series.* (Math. Ann. 115: 581-612, 1938) Young considered more general (strong) Φ-variations.

Let $\mathcal{V}$ denotes the class of all functions $\Phi : [0, \infty) \to [0, \infty)$ which are strictly increasing, continuous, unbounded, and 0 at 0. For any $\Phi \in \mathcal{V}$ we define

$$V^{\Phi}(f, [a, b]) := \sup_n \sup_{a \le x_0 < x_1 < \ldots < x_n \le b} \sum_{i=1}^{n} \Phi\left(\left|f\left(x_i\right) - f\left(x_{i-1}\right)\right|\right).$$

For $\Phi, \Psi \in \mathcal{V}$ let us consider their inverses ϕ and ψ respectively and define

$$\Theta\left(\phi, \psi\right) := \sum_{k=1}^{ns} \phi\left(\frac{1}{k}\right) \psi\left(\frac{1}{k}\right).$$

The generalised Young integral

Theorem (L. C. Young, 1938)

If f and g have no common discontinuities, $V^{\Phi}(f,[a,b]) < +\infty$, $V^{\Psi}(g,[a,b]) < +\infty$ *and* $\Theta(\phi,\psi) < +\infty$ *then the Stieltjes integral* $\int_a^b f(x)\mathrm{d}g(x)$ *exists in the Riemann sense.*

Moreover, for any $\xi \in [a,b]$ *one has the following estimate*

$$\left|\int_a^b f(x)\mathrm{d}g(x) - f(\xi)\left[g(b)-g(a)\right]\right| \leq 20\sum_{k=1}^{\infty}\phi\left(\frac{V^{\Phi}(f,[a,b])}{k}\right)\psi\left(\frac{V^{\Psi}(g,[a,b])}{k}\right).$$

Remark

Leśniewicz and Orlicz showed in 1973 that the series condition $\Theta(\phi,\psi) < +\infty$ *is the best possible for convex functions* Φ *and* Ψ *satisfying the* Δ_2 *and* ∇_d *growth conditions* $(d>1)$: $\Psi(2u) \leq C\Psi(u)$, $2\Psi(u) \leq \Psi(du)/d$.

The proofs of Youngs therems

In all proofs of both Youngs therems found in the literature the Loeve-Young inequality is the most cruicial step.

The original proof of this inequality was elementary but pretty complicated. Later, the proof was simplified. For example, in the book by Terry Lyons and Zhongmin Quian *System Control and Rough Paths*, Oxford Univ. Press 2002, or in lecture notes by Terry Lyons, Michael Caruana and Thierry Levy *Differential Equations Driven by Rough Paths*, Springer 2007, it was based on the approximation of the Riemann sum

$$\sum_{i=1}^{n} f(\xi_i) \{g(x_i) - g(x_{i-1})\}$$

by a sum where one of the numbers $x_1 < \ldots < x_{n-1}$ is excluded from the partition. Iterating this procedure one finds that this sum is sufficiently close to the difference $f(\xi)\{g(b) - g(a)\}$ for some $\xi \in [a, b]$.

Alternative approach

Yet another approach, which gives even better order of the error of the approximation

$$\int_a^b f(x)\mathrm{d}g(x) - f(\xi)\left\{g(b) - g(a)\right\}$$

may be based on the concept of (uniform) approximation of irregular signals f and g with given accuracy $\varepsilon > 0$ by their regularisations - signals with finite total variations.
Let us define

$$\mathrm{TV}^{\varepsilon}(f,[a,b]) := \sup_n \sup_{a \le x_0 < x_1 < \ldots < x_n \le b} \sum_{i=1}^{n} \max\left(\left|f(x_i) - f(x_{i-1})\right| - \varepsilon, 0\right).$$

The quantity $\mathrm{TV}^{\varepsilon}(f,[a,b])$ is called **truncated variation**.

The truncated variation

Theorem (Łochowski, 2013 for cadlag functions; Ghomrasni and Łochowski for regulated functions, 2015)

If $f[a,b] \to \mathbb{R}$ *is regulated then*

$$TV^{\varepsilon}(f,[a,b]) = \inf\left\{ TV(f^{\varepsilon},[a,b]) : \sup_{a\le s\le b}|f(s)-f^{\varepsilon}(s)| \le \frac{\varepsilon}{2}\right\}. \quad (2)$$

Remark

The bound

$$TV^{\varepsilon}(f,[a,b]) \le \inf\left\{ TV(f^{\varepsilon},[a,b]) : \sup_{a\le s\le b}|f(s)-f^{\varepsilon}(s)| \le \frac{\varepsilon}{2}\right\}$$

follows immediately from the estimate: if $\sup_{a\le s\le b}|f(s)-f^{\varepsilon}(s)| \le \frac{\varepsilon}{2}$ *then*

$$\max\left(|f(x_i)-f(x_{i-1})|-\varepsilon,0\right) \le |f^{\varepsilon}(x_i)-f^{\varepsilon}(x_{i-1})|.$$

The truncated variation - remarks

Let $p \geq 1$, $x \geq 0$ and $\varepsilon > 0$. From the inequality

$$\varepsilon^{p-1} \max(x - \varepsilon, 0) \leq x^p$$

we immediately get that whenever $\mathrm{V}^p(f, [a, b]) < +\infty$ then

$$\sup_{\varepsilon > 0} \left\{ \varepsilon^{p-1} \mathrm{TV}^{\varepsilon}(f, [a, b]) \right\} \leq \mathrm{V}^p(f, [a, b])$$

or, equivalently,

$$\mathrm{TV}^{\varepsilon}(f, [a, b]) \leq \frac{\mathrm{V}^p(f, [a, b])}{\varepsilon^{p-1}} \text{ for any } \varepsilon > 0.$$

Thus, for any function with finite (strong) p-variation ($p > 1$) but infinite total variation (corresponding to $p = 1$) we get that its truncated variation $\mathrm{TV}^{\varepsilon}(f, [a, b])$ tends to $+\infty$ no faster than $1/\varepsilon^{p-1}$.

Every such function may be uniformly approximated with the accuracy $\varepsilon > 0$ by a function whose total variation is of order $1/\varepsilon^{p-1}$.

A theorem on the existence of the Riemann-Stieljes integral in terms of the truncated variations

Theorem

Let $f, g : [a; b] \to \mathbb{R}$ *be two regulated functions which have no common points of discontinuity. Let* $\eta_0 \geq \eta_1 \geq \ldots$ *and* $\theta_0 \geq \theta_1 \geq \ldots$ *be two sequences of non-negative numbers, such that* $\eta_k \downarrow 0$, $\theta_k \downarrow 0$ *as* $k \to +\infty$. *Define* $\eta_{-1} := \sup_{a \leq t \leq b} |f(t) - f(a)|$ *and*

$$S := \sum_{k=0}^{+\infty} 2^k \eta_{k-1} \cdot TV^{\theta_k}(g, [a; b]) + \sum_{k=0}^{\infty} 2^k \theta_k \cdot TV^{\eta_k}(f, [a; b]).$$

If $S < +\infty$ *and* $\xi \in [a, b]$ *then the Riemann-Stieltjes integral* $\int_a^b f \,\mathrm{d}g$ *exists and one has the following estimate*

$$\left| \int_a^b f \,\mathrm{d}g - f(\xi) [g(b) - g(a)] \right| \leq 2S.$$

Corollary - an estimate of the Loeve-Young type

Corollary

Let $f, g : [a; b] \to \mathbb{R}$ *be two functions with no common points of discontinuity. If*

$$\sup_{\varepsilon>0} \varepsilon^{p-1} TV^{\varepsilon}(f, [a, b]) < +\infty \text{ and } \sup_{\delta>0} \delta^{q-1} TV^{\delta}(g, [a, b]) < +\infty,$$

where $p > 1$, $q > 1$, $p^{-1} + q^{-1} > 1$, *then the Riemann Stieltjes* $\int_a^b f \mathrm{d}g$ *exists. Moreover, there exist a constant* $C_{p,q}$, *depending on* p *and* q *only, such that for any* $\xi \in [a, b]$

$$\left| \int_a^b f \mathrm{d}g - f(\xi)[g(b) - g(a)] \right| \leq C_{p,q} \|f\|_{p-TV,[a;b]}^{p-p/q} \|f\|_{osc,[a;b]}^{1+p/q-p} \|g\|_{q-TV,[a;b]},$$

where $\|f\|_{p-TV,[a;b]} := \left(\sup_{\varepsilon>0} \varepsilon^{p-1} TV^{\varepsilon}(f, [a, b])\right)^{1/p}$.

Young's theorem in Banach spaces

It is possible to state an analog of the Young theorem and the Loeve-Young inequality for integrals driven by signals attainig their values in some Banach space E.

In such a setting the integral

$$\int_a^b f(x) \mathrm{d}g(x)$$

makes sense if $g : [a, b] \to E$ and $f : [a, b] \to L(E, V)$, where V is another Banach space and $L(E, V)$ denotes the space of (continuous) linear mapings from E to V, $F : E \to V$, with the usual norm

$$\|F\|_{L(E,V)} = \sup_{u \in E, \|u\|_E = 1} Fu.$$

To prove the existence of this integral using truncated variation techniques one needs to prove an analog of relation (2) in higher dimensional spaces.

The truncated variation in higher dimensions - properties

If $f : [a, b] \to E$ attains its values in a metric space E with the metric d then we may define

$$\mathrm{TV}^{\varepsilon}(f, [a, b]) := \sup_{n} \sup_{a \le x_0 < x_1 < \ldots < x_n \le b} \sum_{i=1}^{n} \max \{ d(f(x_i), f(x_{i-1})) - \varepsilon, 0 \}.$$

Unfortunately, even for $E = \mathbb{R}^2$ relation (2) is no longer valid. However, for any regulated $f : [a, b] \to E$ we have the following estimate:

$$\mathrm{TV}^{\varepsilon}(f, [a, b]) \le \inf_{f^{\varepsilon} \in B(f, \varepsilon/2)} \mathrm{TV}(f^{\varepsilon}, [a, b]) \le 2 \cdot \mathrm{TV}^{\varepsilon/4}(f, [a, b]),$$

where

$$B(f, \varepsilon/2) := \left\{ g : d_{\infty,[a,b]}(f, g) \le \varepsilon/2 \right\}$$

denotes the ball with the center at f and the radius $\varepsilon/2$, in the supremum metric $d_{\infty,[a,b]}(f, g) := \sup_{s \in [a,b]} d(f(s), g(s))$.

A theorem on the truncated variation in higher dimensions

Theorem (Lochowski 2016)

For any regulated $f : [a,b] \to E$ *there exists a step function* $f^c : [a,b] \to E$ *such that* $\sup_{t\in[a,b]} d\left(f(t), f^c(t)\right) \le c/2$ *and for any* $\lambda > 1$, $TV(f^c,[a,b]) \le \lambda \cdot TV^{(\lambda-1)c/(2\lambda)}(f,[a,b])$. *Thus the following estimates hold*

$$TV^c(f,[a,b]) \le \inf_{g\in B(f,c/2)} TV(g,[a,b]) \le \inf_{\lambda>1} \lambda \cdot TV^{(\lambda-1)c/(2\lambda)}(f,[a,b]).$$

Moreover, if E *is a vector normed space with the norm* $\|\cdot\|_E$ *then there exists* $f^{c,lin} : [a,b] \to E$ *such that* $f^{c,lin}$ *is piecewise linear, jumps of* $f^{c,lin}$ *occur only at the points where the jumps of* f *occur,* $\sup_{t\in[a,b]} \left\|f(t) - f^{c,lin}(t)\right\|_E \le c$ *and* $TV\left(f^{c,lin},[a,b]\right) = TV(f^c,[a,b])$.

A theorem on the existence of the Riemann-Stieltjes integral in higher dimensions

Theorem

Let $f : [a, b] \to L(E, V)$ *and* $g : [a, b] \to E$ *be two regulated functions which have no common points of discontinuity. Let* $\eta_0 \geq \eta_1 \geq \ldots$ *and* $\theta_0 \geq \theta_1 \geq \ldots$ *be two sequences of positive numbers, such that* $\eta_k \downarrow 0$, $\theta_k \downarrow 0$ *as* $k \to +\infty$. *Define* $\eta_{-1} := \frac{1}{2} \sup_{a \leq t \leq b} \|f(t) - f(a)\|_{L(E,V)}$ *and*

$$S := 4 \sum_{k=0}^{+\infty} 3^k \eta_{k-1} \cdot TV^{\theta_k/4}(g, [a, b]) + 4 \sum_{k=0}^{\infty} 3^k \theta_k \cdot TV^{\eta_k/4}(f, [a, b]).$$

If $S < +\infty$ *then the Rieman-Stieltjes integral* $\int_a^b f \,\mathrm{d}g$ *exists and for any* $\xi \in [a, b]$ *one has the following estimate*

$$\left\| \int_a^b f \,\mathrm{d}g - f(\xi)[g(b) - g(a)] \right\|_V \leq 2S. \qquad (3)$$

An estimate of the Loeve-Young type in Banach spaces

Corollary

Let $f : [a, b] \to L(E, V)$, $g : [a, b] \to E$ be two functions with no common points of discontinuity. If

$$\sup_{\varepsilon>0} \varepsilon^{p-1} TV^{\epsilon}(f, [a, b]) < +\infty \text{ and } \sup_{\delta>0} \delta^{q-1} TV^{\delta}(g, [a, b]) < +\infty,$$

where $p > 1$, $q > 1$, $p^{-1} + q^{-1} > 1$, then the Riemann Stieltjes $\int_a^b f \, \mathrm{d}g$ exists. Moreover, there exist a constant $D_{p,q}$, depending on p and q only, such that for any $\xi \in [a, b]$

$$\left| \int_a^b f \, \mathrm{d}g - f(\xi) [g(b) - g(a)] \right| \\ \leq D_{p,q} \|f\|_{p-TV,[a;b]}^{p-p/q} \|f\|_{osc,[a;b]}^{1+p/q-p} \|g\|_{q-TV,[a;b]},$$

where $\|f\|_{p-TV,[a;b]} := \left(\sup_{\varepsilon>0} \varepsilon^{p-1} TV^{\epsilon}(f, [a, b])\right)^{1/p}$.

Problem with the definition of integrals driven by more irregular paths

All the results mentioned so far are insufficient to define for example the integral

$$\int_0^T B_t \mathrm{d} B_t,$$

where B_t is a standard Brownian motion - a stochastic process widely used in modelling the evolution of stock prices, stock indices etc.

It is well known that $Ind_{\mathrm{var}}\left(B, [0, T]\right) = 2$, which makes it impossible to apply Young's theorem in this case.

A real breakthrough in the understanding of this difficulcy was provided by the **rough paths theory**, developed mainly by Terry Lyons in early 90s of the last century.

Multiplicative functionals

Assume that V is a Banach space with norm $\|\cdot\|$ and we are given a sequence of tensor products $V^{\otimes k} = V \otimes V \otimes \ldots \otimes V$ (k copies of V) together with norms $\|\cdot\|_k$ on $V^{\otimes k}$ satisfying $\|\xi \otimes \eta\|_{k+l} \leq \|\xi\|_k \|\eta\|_l$ for $\xi \in V^{\otimes k}$ and $\eta \in V^{\otimes l}$. For each $n = 1, 2 \ldots$ we build the following (truncated) tensor algebra $T^{(n)}V$:

$$T^{(n)}V = \sum_{k=0}^{n} \oplus V^{\otimes k}, \quad V^{\otimes 0} = \mathbb{R}$$

with multiplication (tensor product) being the usual multiplication as polynomials, except that the higher-order (than degree n) terms are omitted.

Let now $\Delta := \{(s,t) : 0 \leq s \leq t \leq T\}$. $X : \Delta \to T^{(n)}V$, $X_{s,t} = \left(X_{s,t}^{(0)}, X_{s,t}^{(1)}, \ldots, X_{s,t}^{(n)}\right)$ is called a **multiplicative functional** if $X_{s,t}^{(0)} = 1$ and

$$X_{s,t} \otimes X_{t,u} = X_{s,u}, \text{ for any } (s,t), (t,u) \in \Delta. \tag{4}$$

Abstract iterated integrals

Equality (4) is called Chen's identity and it corresponds to the additive property of increments of iterated integrals over concatenated intervals. For example, if $x : [0, T] \to \mathbb{R}$ is a continuous path with finite total variation and we define

$$x_{s,t}^{(1)} = \int_s^t \mathrm{d}x_v \text{ and } x_{s,t}^{(2)} = \int_{s \le v_1 < v_2 \le t} \mathrm{d}x_{v_1} \mathrm{d}x_{v_2}$$

then we get

$$x_{s,u}^{(2)} = x_{s,t}^{(1)} x_{t,u}^{(1)} + x_{s,t}^{(2)} + x_{t,u}^{(2)} \text{ for any } (s,t),(t,u) \in \Delta.$$

In this sense, ith component of a multiplicative functional may be treated as an abstract iterated integral of order i.

Controls and Lyons Extension Theorem

A **control** ω is a continuous, super-additive function on the simplex Δ with values in $[0; +\infty)$ such that $\omega(s,s) = 0$. The superadditivity of ω means that

$$\omega(s,t) + \omega(t,u) \leq \omega(s,u) \text{ for any } 0 \leq s \leq t \leq u \leq T.$$

One says that a map $X : \Delta \to T^{(n)}V$ **possesses finite p-variation** $(p \geq 1)$ if

$$\|X^{(i)}_{s,t}\|_i \leq \omega(s,t)^{i/p} \text{ for } i = 1, 2, \ldots, n; (s,t) \in \Delta.$$

Theorem (Lyons Extension theorem)

If $X : \Delta \to T^{(n)}V$ is a multiplicative functional with finite p-variation and $n \geq \lfloor p \rfloor$ then all abstract iterated integrals of order greater than n may be calculated with the integrals of order $1, 2, \ldots, n$ which means simply that for any $m > n$ the functional X may be uniquely extended to a multiplicative functional $\tilde{X} : \Delta \to T^{(m)}V$ with finite p-variation, which agrees with X on first n components.

Ambiguity of iterated integrals driven by signals with infinite p-variation

The simplest non-trivial case of Lyons Extension Theorem may be viewed as an abstract formulation of Young's theorem for $p = q \in (1, 2)$, since for such p and q, $1/p + 1/q = 2/p > 1$ and thus, Young's theorem guarantees the existence of $x^{(2)}_{s,t} = \int_{s \le v_1 < v_2 \le t} \mathrm{d}x_{v_1} \mathrm{d}x_{v_2}$.

It is well known that for signals with higher degree of irregularity, with infinite p-variation for all $p < 2$ in general one can not uniquely construct integrals of the form $\int_{s \le v_1 < v_2 \le t} \mathrm{d}X_{v_1} \mathrm{d}Y_{v_2}$ based on the first order integrals $\int_{s \le v \le t} \mathrm{d}X_v$, $\int_{s \le v \le t} \mathrm{d}Y_v$.

For example, if $\left(1, X^{(1)}, X^{(2)}\right)$ is a multiplicative functional with finite 2-variation, then for any bounded linear operator $A \in V^{\otimes 2}$, $\left(1, X^{(1)}, X^{(2)} + (t - s)A\right)$ is another multiplicative functional with finite 2-variation.

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Thank you for your attention!