Stock returns modelling– from classical model to models with jumps and extreme events

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Uncertain behaviour of stock returns

- The price of a stock (for example the price of one share of Cemex noted on ***Bolsa Mexicana de Valores***) is not known in advance
- S_t - price today (or at time t)
- S_{t+1} - price tomorrow (or at time t+1 in the future)
- **Return** – how many pesos more will produce tommorow (or in the future – at time t+1) one peso invested in a Cemex share today (at time t)?

Return cont.

- If we had invested K pesos then we could buy K / S_t shares
- If we sell them at time t+1, for S_{t+1} pesos each, then we will get $S_{t+1} K / S_t$ pesos
- Thus one peso produces

$$\frac{S_{t+1} K / S_t}{K} - 1 = \frac{S_{t+1}}{S_t} - 1$$

pesos more (if it is negative, then it means that we have lost some pesos ;-( )

Return cont.

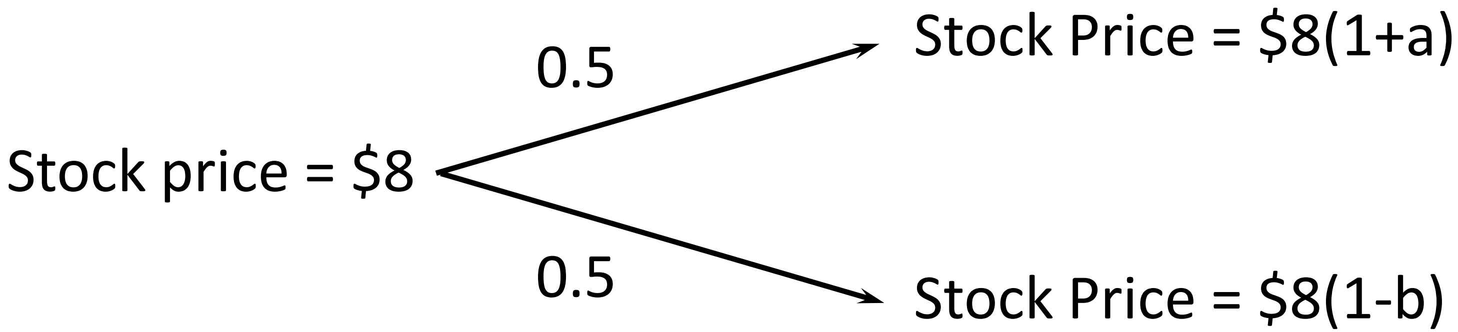
- The return between times t and t+1 reads as

$$R_{t,t+1} = \frac{S_{t+1}}{S_t} - 1$$

- since S_{t+1} is **unknown** the **return is unknown too**
- Even if it is unknown, one may have some expectation about the future price – based on up-to-date observations

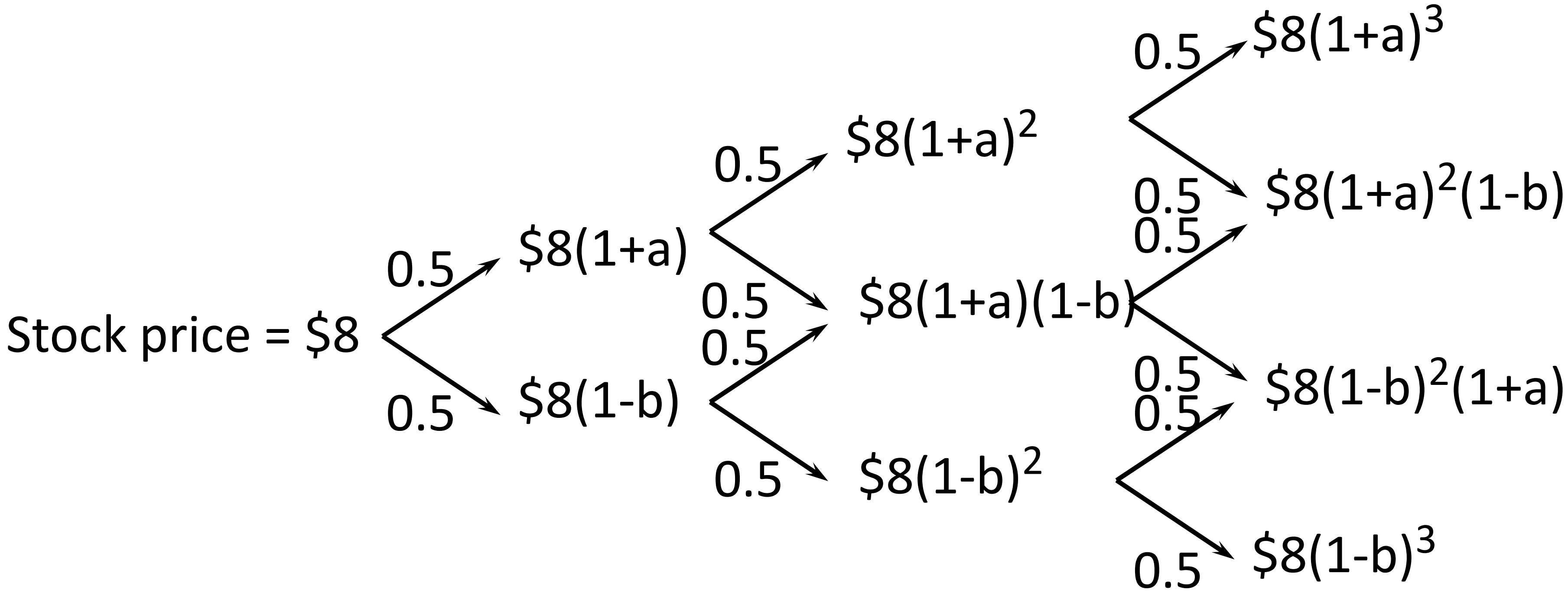
Randomness – the simplest model

- The most simple model assumes that
  - with probability 0.5 the return will be equal a
  - with probability 0.5 the return will be equal -b



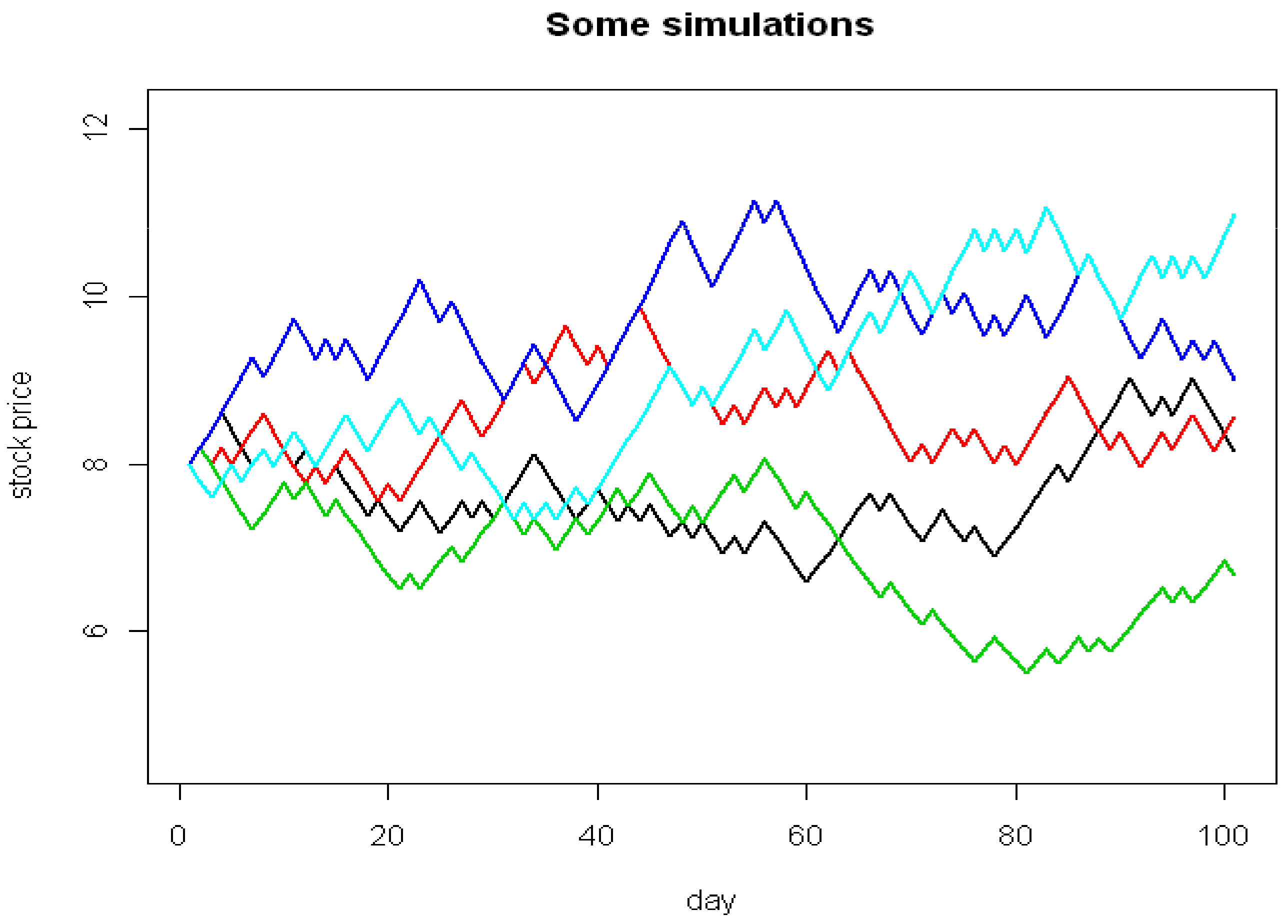
The simplest model cont.

- What will be the stock price after 3 days?
- Possible scenarios



The simplest model cont.

- What will be the stock price after n days?



Real prices


Mar 2, 2011 : CX 8.84
11
10.50
10
9.50
9
8.50
8
© 2011 Yahoo! Inc.
2010
Nov
Dec
2011
Feb
Mar
Volume 9,597,400
60M
40M
20M
1D
5D
1M
YTD
3M
6M
1Y
2Y
5Y
Max
FROM: Sep 23 2010
TO: Mar 22 2011

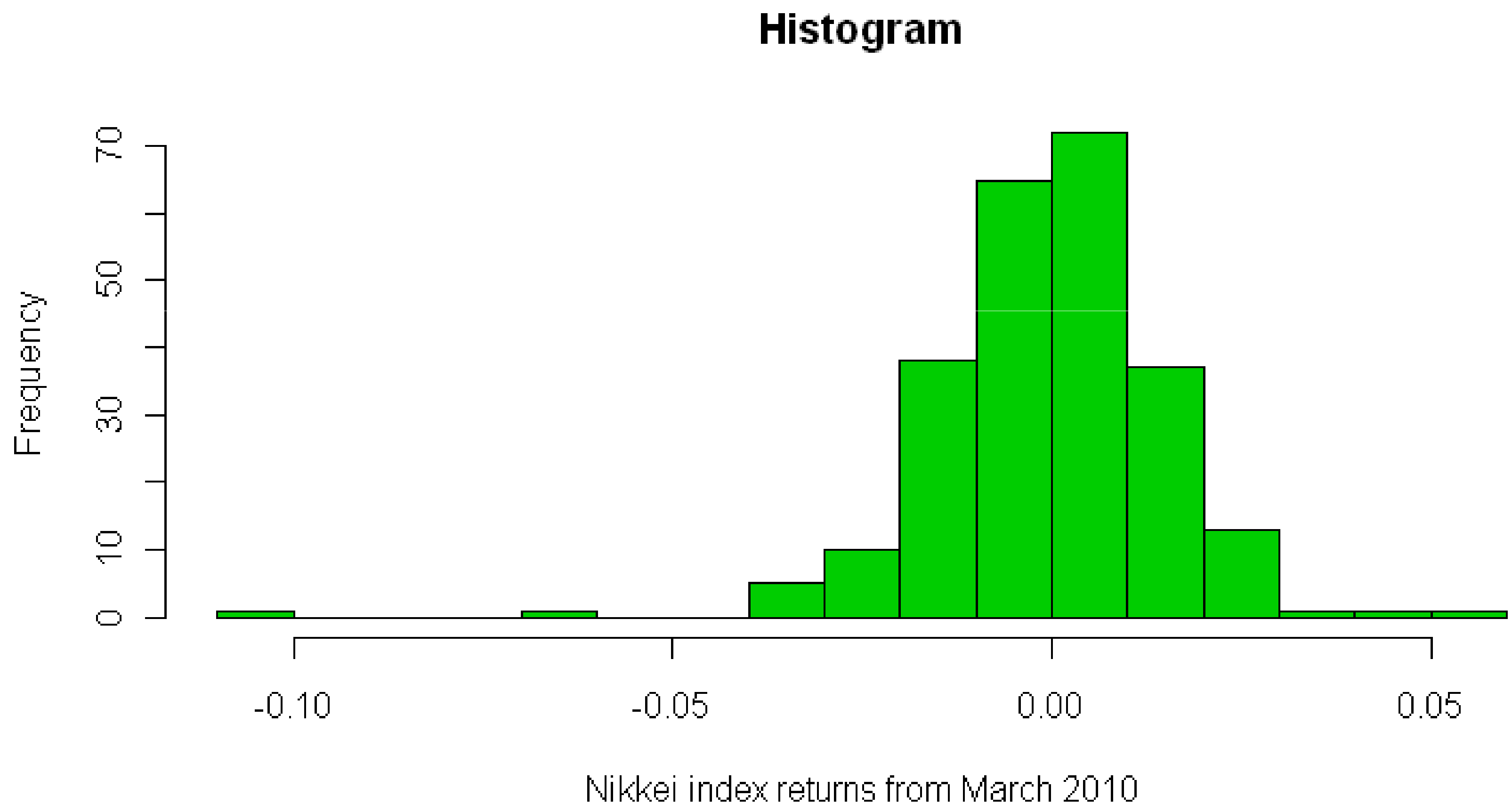
The simplest model vs. real prices

- At the first sight the simplest model seems to provide good approximation of real prices movements
  - the only problem is the choice of the parameters a and b
- But in reality the returns between consecutive days are not always equal +a or –b for some a and b
- They attain a whole spectrum of values

Example – recent returns of Nikkei Index

Nikkei Index					
Date	**Last**	**Open**	**High**	**Low**	**Return %**
03/23/2011	9449,47	9553	9568	9376,5	**-1,65%**
03/22/2011	9608,32	9545	9610,5	9455,5	**4,36%**
03/18/2011	9206,75	9083	9260,5	9075	**2,72%**
03/17/2011	8962,67	8543	9080,5	8443	**-1,44%**
03/16/2011	9093,72	9105	9135,5	8782,5	**5,07%**
03/15/2011	8655	9235	9287,5	7862,5	**-10,04%**
03/14/2011	9620,49	9575	9817,5	9520	**-6,18%**
03/11/2011	10254,43	10350	10375,5	10225,5	**-1,72%**
03/10/2011	10434,38	10528	10543,5	10400,5	**-1,46%**
03/09/2011	10589,5	10615	10660,5	10565,5	**0,61%**
03/08/2011	10525,19	10520	10565,5	10505,5	**0,19%**
03/07/2011	10505,02	10610	10615	10465,5	**-1,76%**

Closer look at the frequency of daily returns od Nikkei Index since March 2010



The need of the refinement of the model

- The need of the refinement follows from significant frequency of the whole spectrum of returns
- The idea – what if we change the prices more frequently, but follow the same pattern?
- In fact **the changes of stock and index prices occur every hour or even every minute**!

Classical model - refinemend of the simplest model

- A bit more sophisticated mathematical considerations lead to the following **classical model**
- Natural logarithm

$$\ln\left(1 + R_{t,t+1}\right)$$

(sometimes called **logarithmic return**) is a **random quantity** (mathematicians say: **random variable**) with **normal distribution**

Normal distribution

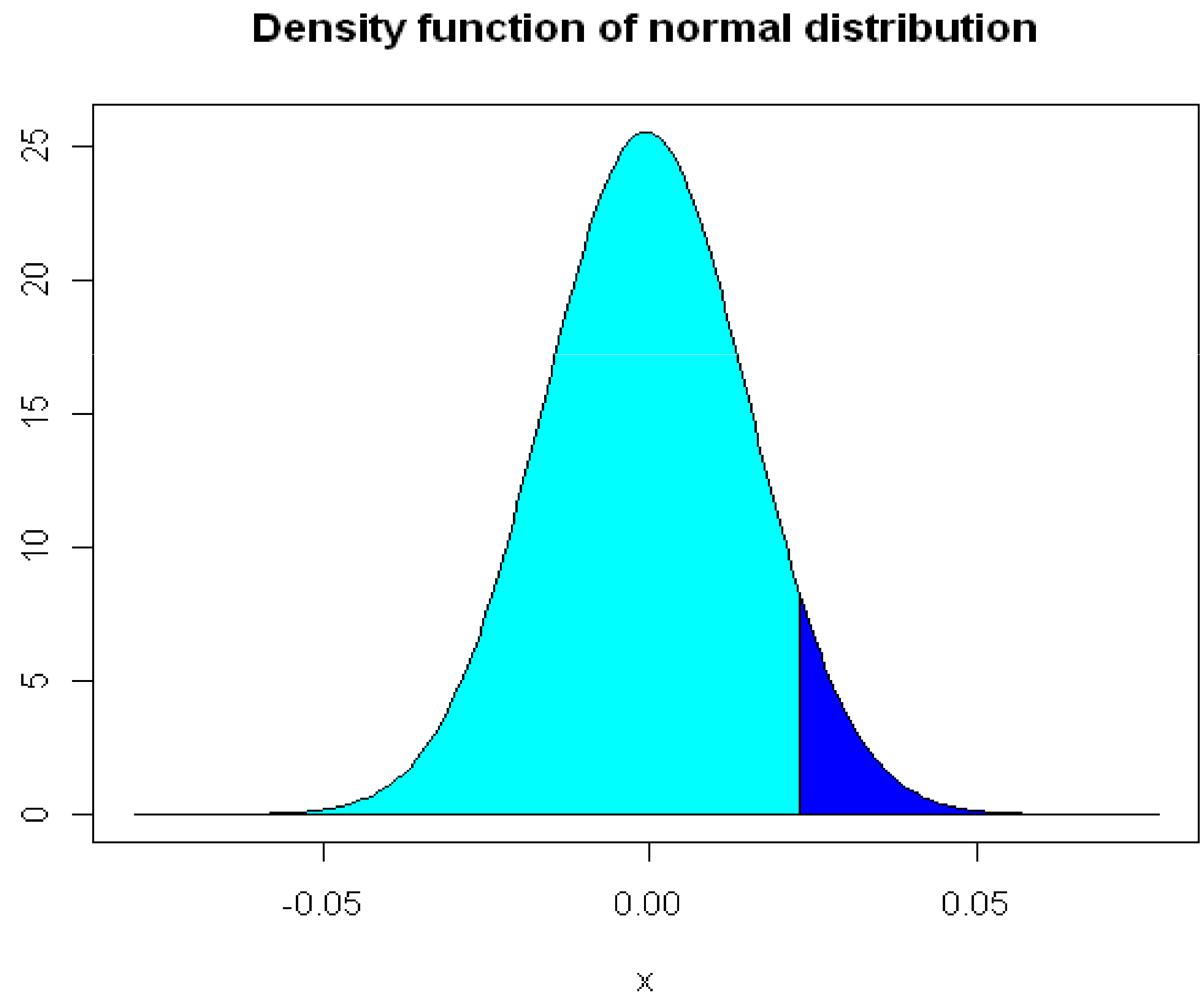
- Normal distribution has two parameters – **mean** and **variance** – and knowing them **we are able to calculate probability** that

  - the daily return will be higher than a certain threshold

  or

  - the daily return will be lower than a certain threshold

Normal distribution - example



Normal distribution - probabilities

- If $\ln\left(1+R_{t,t+1}\right)$ has mean μ and variance σ^2 than the probability that $R_{t,t+1}$ will be smaller than some R may be calculated with a bit complicated formula

$$P\left(R_{t,t+1} \leq R\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\ln(1+R)} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

- but it is simply the area of the light-blue figure on the previous slide

Calculation of probabilities in the classical model

- For daily returns (since March 2010) of the Nikkei Index we have estimated

$$\mu \approx -0.0005, \sigma \approx 0.016$$

- and calculated the probabilities

$$P\left(R_{t,t+1} \le \mu - 4\sigma\right) = P\left(R_{t,t+1} \ge \mu + 4\sigma\right) \approx 0.00003$$

$$P\left(R_{t,t+1} \le \mu - 3\sigma\right) = P\left(R_{t,t+1} \ge \mu + 3\sigma\right) \approx 0.00135$$

$$P\left(R_{t,t+1} \le \mu - 2\sigma\right) = P\left(R_{t,t+1} \ge \mu + 2\sigma\right) \approx 0.02275$$

$$P\left(R_{t,t+1} \le \mu - \sigma\right) = P\left(R_{t,t+1} \ge \mu + \sigma\right) \approx 0.15865$$

Probabilities as the measures of risk

- The knowledge of the calculated probabilities is important in risk management
- They tell a stockholder **for what loss he/she shall be prepared to**
- Since they are important they have even their own name – Value at Risk
- In classical model they depend only on the parameters μ and σ

Probabilities - estimation

- Since there were approximately 250 trading days on Tokio Stock Exchange, the **probability that the daily return will be higher than a certain threshold** R may be estimated as

$$\frac{number\ of\ days\ with\ biger\ return\ than\ R}{250}$$

- similarly, the probability that the daily return will be lower than a certain threshold R reads

$$\frac{number\ of\ days\ with\ smaller\ return\ than\ R}{250}$$

Probabilities - comparison

Threshold R	Theoretical probability return <R	Number of days return<R	Estimated probability return<R
-6,23%	0,00003	1	0,00408
-4,68%	0,00135	2	0,00816
-3,14%	0,02275	5	0,02041
-1,59%	0,15866	32	0,13061

Threshold R	Theoretical probability return >R	Number of days return>R	Estimated probability return>R
6,23%	0,00003	0	0,00000
4,68%	0,00135	1	0,00408
3,14%	0,02275	3	0,01224
1,59%	0,15866	28	0,11429

Problem with the classical model – extreme events

- The classical model poorly predicts **occurence of extreme events** (such as the occurence of the daily return of Nikkei Index lower than -6% - due to the earthquake and tsunami in Japan)
- Such events occur more often than classical model predicts, and what is even more problematic – **they may cause much bigger loses than „usual” events**

Extreme events – help!!!

- Great help in modelling extreme events provides to us modern **Extreme Value Theory (EVT)**
- It gives general framework to estimate **probabilities of extreme values**
- The idea is to choose a certain thresholds $R_- < 0, R_+ > 0$ and to model separately returns
  - between R_- and R_+ – with classical model
  - smaller than R_- or greater than R_+ – with EVT

Probabilities of extreme values

- The formula for probability of extreme return, greater than $R_+ + u$, given by EVT reads as the following product

$$P\left(R_{t,t+1} \geq R_+ + u\right) = \left(1 + \frac{\xi_+ u}{\beta_+}\right)^{-1/\xi_+} \times P\left(R_{t,t+1} \geq R_+\right)$$

- where probability $P\left(R_{t,t+1} \geq R_+\right)$ is calculated with classical model
- Similarly we deal with left tail probabilities, i.e.

$$P\left(R_{t,t+1} \leq R_- - u\right)$$

EVT – what we need to use it?

- To use EVT we need to fix here three more parameters: $R_+ > 0, \xi_+ > 0, \beta_+ > 0$ - threshold, shape and scale parameters
- Similarly, we need three more parameters to calculate left tail probabilities
- It may be a bit difficult, because we have small number of observations exceeding thresholds
- But once it is done, we have much realistic prediction of probabilities of extreme values

EVT for Nikkei Index

- For Nikkei Index we obtain the following estimates based on 10 years observations

$$P\left(R_{t,t+1} \geq 4\% + u \mid R_{t,t+1} \geq 4\%\right) \approx \left(1 + 25 \cdot u\right)^{-4}$$

- Below we have comparison of the probabilities calculated with this formula and real frequencies

u	real frequency R>4%+u among all 34 returns >4%	calculated frequency R>4%+u among all 34 returns >4%
0,50%	0.6	0,624295077
1%	0.5	0,4096
1,50%	0.32	0,279762311
2%	0.2	0,197530864
3%	0.15	0,106622241

EVT for Nikkei Index, cont.

- For risk measurement it is more important to calculate probabilities of the adverse market movements, and they are associated with negative returns. We have

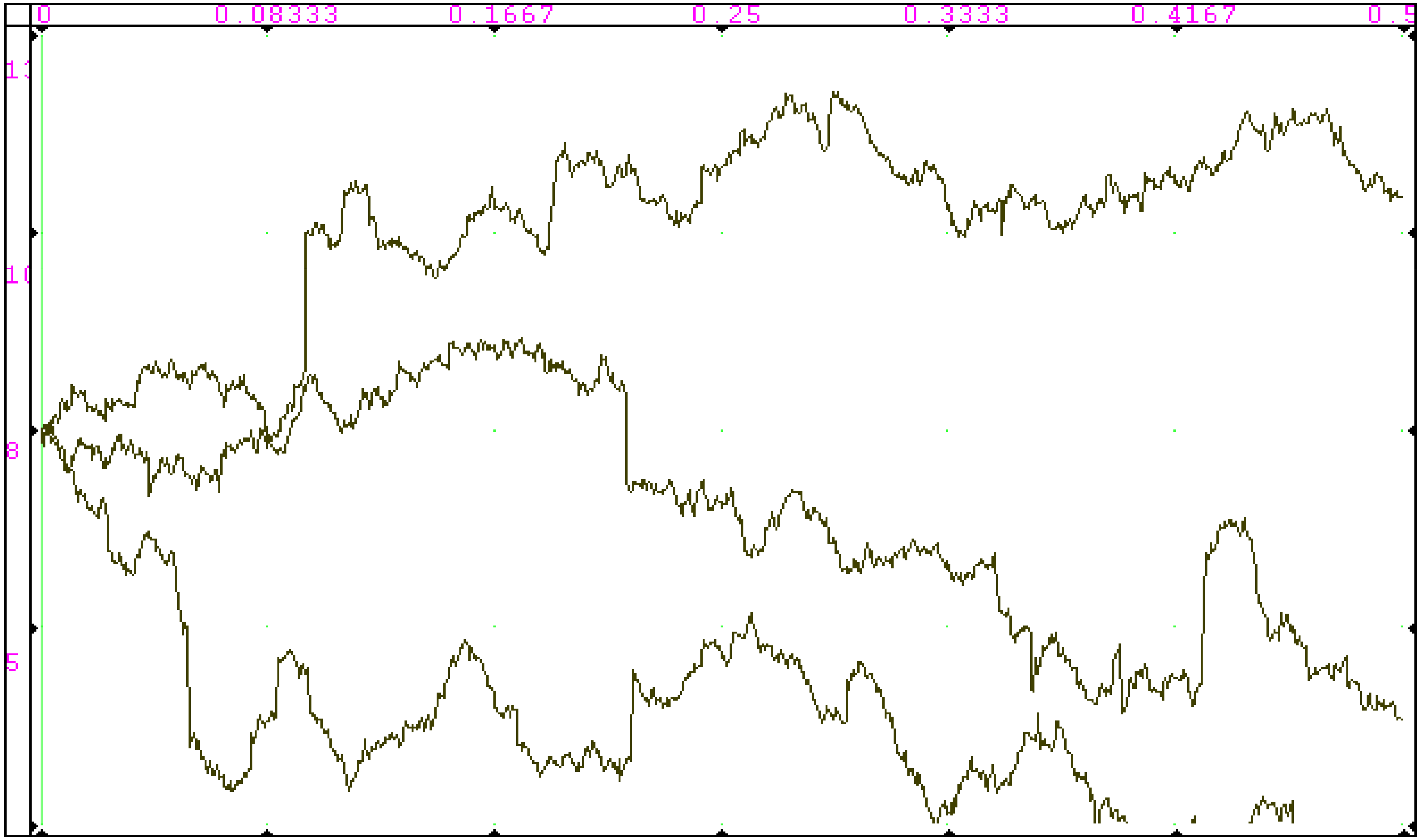
$$P\left(R_{t,t+1} \leq -2\% - u \mid R_{t,t+1} \leq -2\%\right) \approx \left(1 + 28.75 \cdot u\right)^{-4.35}$$

u	real frequency R<-2%-u among all 310 returns <-2%	calculated frequency R<-2%-u among all returns <-2%
0,50%	0.545	0,557519015
1%	0.335	0,333119086
1,50%	0.2	0,210202355
2%	0.145	0,138621174
3%	0.087	0,066846885
4%	0.038	0,035800797
5%	0.0226	0,020739147
6%	0.0129	0,012768965

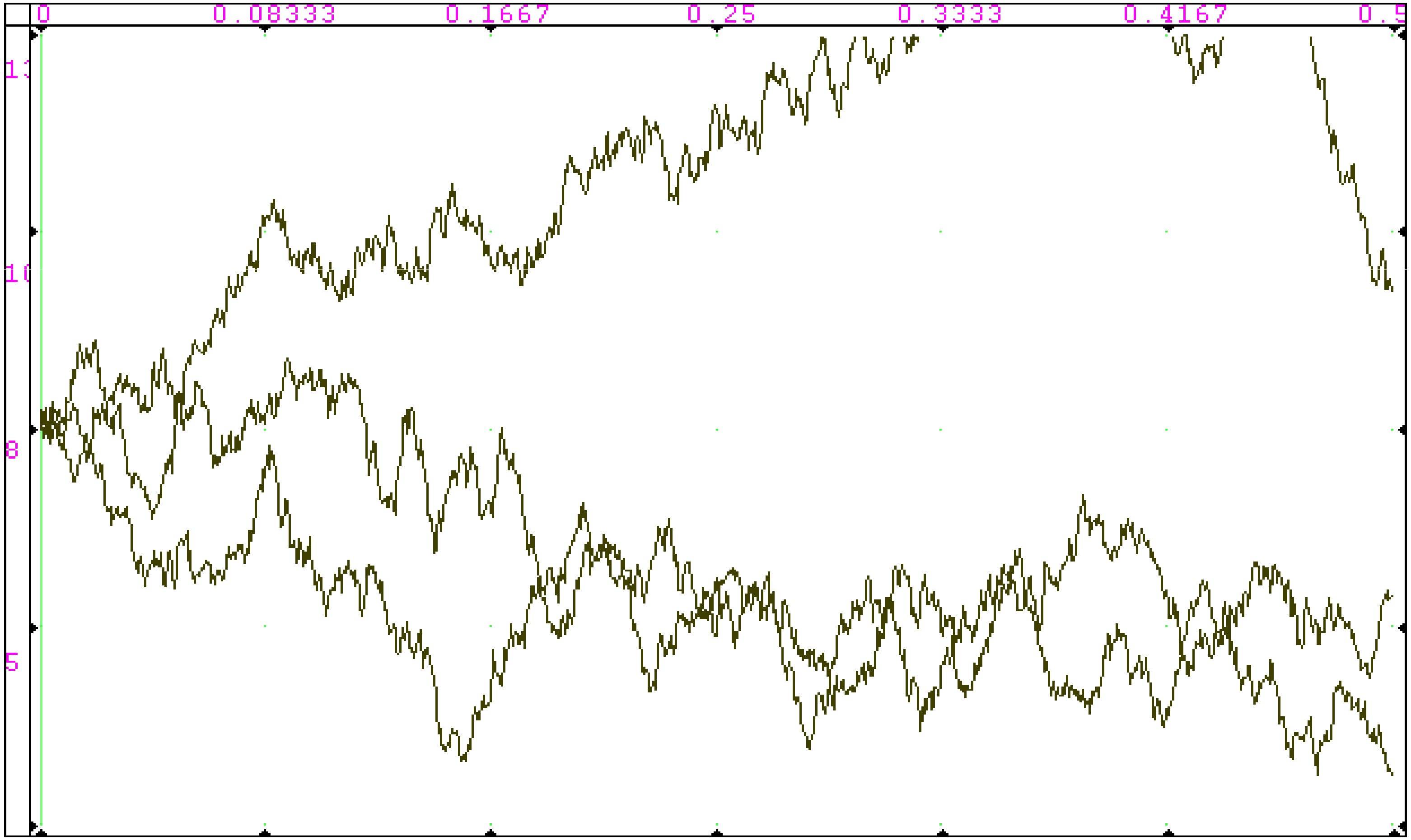
Other models – models with jumps

- There are other approaches possible to deal with extreme events – one of them is to allow jumps in simulations of stock prices
- This approach uses more advanced mathematical models
- They may seem difficult at the first sight but the mathematics in their background is very beautiful

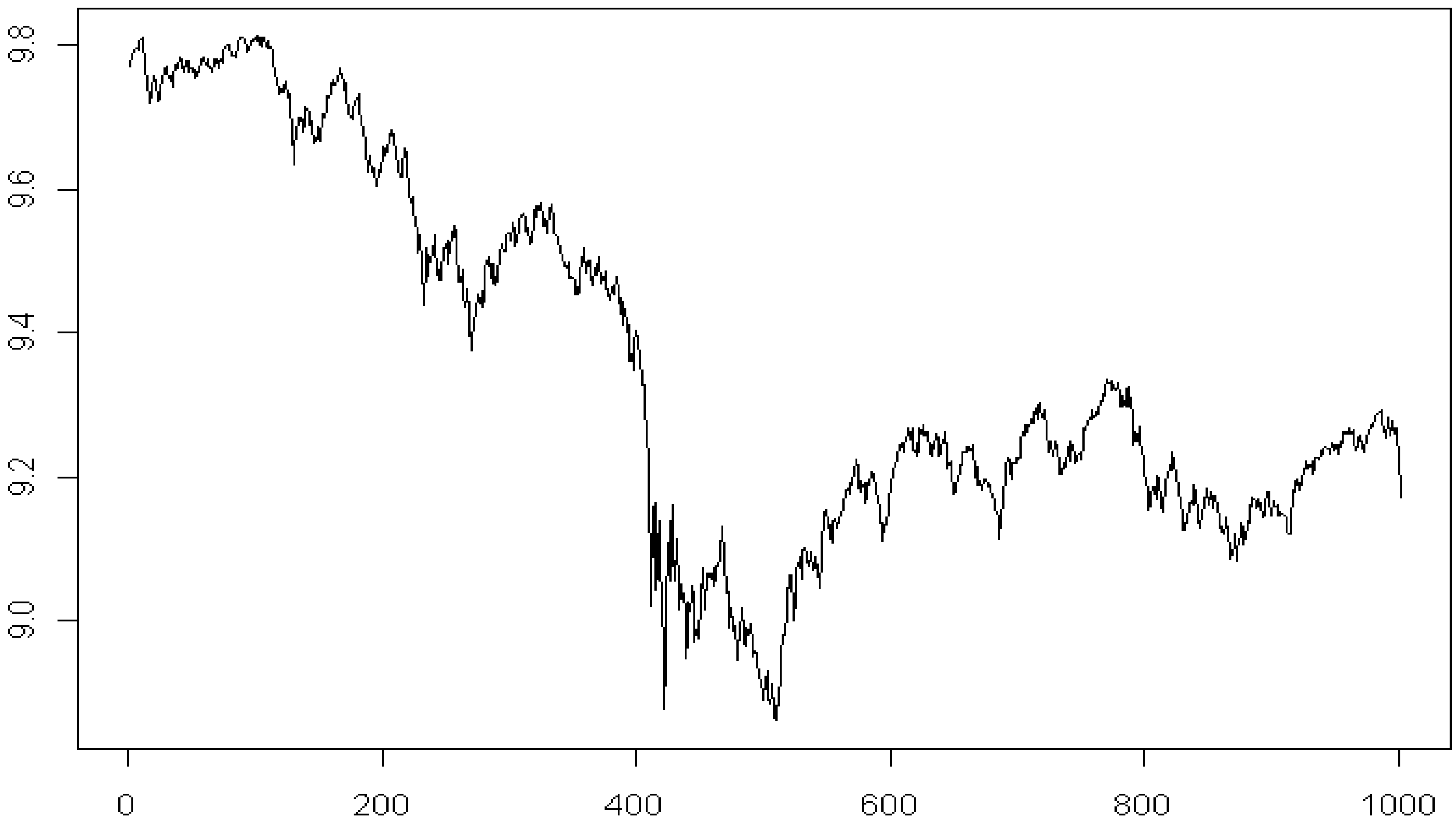
Model with jumps



Classical model



Real quotes



Thank you!