Surplus process and ruin theory

Risk theory
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Compound Poisson distribution as a distribution of total claim amount

- \( Y \) - size of a (typical) claim in insurer’s portfolio
- \( Y, Y_1, Y_2, \ldots \) - i.i.d. random variables
- \( N_t \) - homogeneous Poisson process with intensity \( \lambda \), modelling claims’ arrival (number of claims between 0 and \( t \))
- Total claim amount on time interval \([0, t]\) may be written as
  \[
  S_t = Y_1 + Y_2 + \ldots + Y_{N_t}
  \]
Surplus process

- Surplus process (or risk process) is a model of accumulation of insurer’s capital
- $u$ - the initial capital;
- $c$ - the (constant) premium income per unit time;
- **Surplus process** is defined as

\[ U_t = u + c \cdot t - S_t, \quad t \geq 0 \]
Moment of a ruin and ruin probability

• When surplus process attains negative value it means, that the claims exceed initial capital and the income from premiums

• **Moment of a ruin** is defined as

\[ T = \inf \{ t \geq 0 : U_t < 0 \} \]

• **Ruin probability** before moment \( t \) is defined as

\[ \psi (u, t) = P (T < t) \]
Surplus process in discrete time

- When surplus process is observed in discrete moments $t = 0, 1, 2, 3, \ldots$, we call it a **surplus process in discrete time**

$$U_n = u + c \cdot n - S_n, \ n = 0, 1, 2, \ldots$$

- For $c \leq \mathbb{E} S_1$, from law of large numbers we get that almost surely

$$\lim_{n \to \infty} \left( U_n / n \right) = c - \mathbb{E} S_1 \leq 0$$

- from which follows

$$\psi(u) := \psi(u, +\infty) = \mathbb{P}(T < \infty) = 1$$
Loading (or safety) factor

• For $c > \mathbb{E}S_1$ for some $\theta > 0$ we have
  \[ c = (1 + \theta)\mathbb{E}S_1 \]

• The coefficient $\theta > 0$ is called \textit{loading (or safety) factor}

• Since $S_1$ has compound Poisson distribution, we get
  \[ \theta = \frac{c}{\mathbb{E}S_1} - 1 = \frac{c}{\lambda \mathbb{E}Y} - 1 \]

• It is possible to prove that
  \[ \psi(0) = 1 / (1 + \theta) \]
Adjustment coefficient

• For a positive safety factor, there exists exactly one positive solution (in R) of the equation

\[ e^{Rc} = M_{S_1}(R) = \mathbf{E}e^{RS_1} \]

which is called **adjustment coefficient**

• It may be proved that

\[ \lambda + cR = \lambda M_Y(R) = \lambda \mathbf{E}e^{RY} \]
Adjustment coefficient, cont.

- Problem (theoretical one): to prove statement from the previous slide, i.e. that for the positive loading factor, adjustment coefficient exists and is uniquely determined

- Problem (computational): calculate the adjustment coefficient in the special case when claim sizes have exponential distribution with parameter $\beta$
Lundberg inequality

• The following inequality is valid \[ \psi (u) \leq e^{-R \cdot u} \]
• Proof: let \[ dP (y) = P \left( Y \in \left[ y, y + dy \right) \right) \]
and \( \psi_n (u) \) denote, that the ruin has occurred before \( n+1 \)th claim and that \( \psi_{n-1} (u) \leq e^{-R \cdot u} \)
then
\[ \psi_n (u) = \int_0^\infty \int_0^\infty \psi_{n-1} (u + ct - y) dP (y) \lambda e^{-\lambda t} dt \]
\[ \leq \int_0^\infty \int_0^\infty \exp \left( -R (u + ct - y) \right) dP (y) \lambda e^{-\lambda t} dt \]
\[ = e^{-R \cdot u} \frac{\lambda}{\lambda + cR} E e^{y \cdot R} = e^{-R \cdot u} \]
Exact theoretical formula for ruin probability

• For every time moment $t$ we have

$$E e^{-R \cdot U_t} = E e^{-R(u + ct + S_t)} = e^{-Ru - Rct} E e^{R \cdot S_t}$$

$$= e^{-Ru} e^{-Rct} e^{\lambda t(M_Y(R) - 1)} = e^{-Ru} \left(e^{-Rc - \lambda + \lambda M_Y(R)}\right)^t$$

$$= e^{-Ru}$$

• and the following exact formula holds

$$E \left[ e^{-RUT} \mid T < \infty \right] \cdot \psi(u) = e^{-R \cdot u}$$
Functional equation for ruin probability

- For $u<0$ let us define $\psi(u) = 1$
- Then we have the following formula
  $$\psi(u) = \int_0^\infty \int_0^\infty \psi(u + ct - y) dP(y) \lambda e^{-\lambda t} dt$$
- From the above formula we derive (a bit complicated) equation
  $$\frac{C}{\lambda} \psi'(u) = \psi(u) - \int_0^\infty \psi(u - y) dP(y)$$
  $$= \psi(u) - \int_0^u \psi(u - y) dP(y) - P(Y > u)$$

Problem (theoretical): prove the above equation
Ruin probability for exponential claim sizes

• With \( d\mathbf{P}(y) = \beta e^{-\beta y} dy = p(y) dy \), differentiating the equation (星空) we get

\[
\frac{c}{\lambda} \psi''(u) = \psi'(u) - p(0) \psi(u)
\]

\[
+ \beta \int_{0}^{u} \psi(z) p(x - z) dz + p(u)
\]

• Now, summing it with equation (星空) we have

\[
\frac{c}{\lambda} \psi''(u) + \theta \psi'(u) = 0
\]