Truncated variation - its properties and applications in stochastic analysis

Rafał M. Łochowski

Osaka 2015

Outline

1. Definition and basic properites
2. The truncated variation as a measure of path irregularity
3. Realized volatility estimation
4. Limit theorems for segment crossings

Truncated variation - definition

For a path $f : [a; b] \to E$, where E is a normed space with norm $|\cdot|$, its **truncated variation** is defined with the following formula

$$\mathrm{TV}^c(f,[a;b]) := \sup_n \sup_{a \le t_1 < t_2 < \ldots < t_n \le b} \sum_{i=1}^{n-1} \max\left\{|f(t_{i+1}) - f(t_i)| - c, 0\right\}, \tag{1}$$

where $c > 0$ is the truncation parameter.

Fact

If E is complete (i.e. Banach) space, then $TV^c(f,[a;b]) < +\infty$ for any $c > 0$ iff f is regulated, i.e. it has right and left limits.

Truncated variation - basic properties

Let $\|f-g\|_\infty := \sup_{t\in[a;b]} |f(t)-g(t)|$.
From the triangle inequality, for g such that $\|f-g\|_\infty \le c/2$, we immediately get

$$|f(t_{i+1})-f(t_i)|-c \le |g(t_{i+1})-g(t_i)|$$

and from this

$$\mathrm{TV}^c(f,[a;b]) \le \mathrm{TV}(g,[a;b]), \tag{2}$$

where $\mathrm{TV}(g,[a;b])$ is just the (total) variation of g.

Truncated variation - basic properties

Let $\|f-g\|_\infty := \sup_{t\in[a;b]} |f(t)-g(t)|$.
From the triangle inequality, for g such that $\|f-g\|_\infty \le c/2$, we immediately get

$$|f(t_{i+1})-f(t_i)| - c \le |g(t_{i+1})-g(t_i)|$$

and from this

$$\mathrm{TV}^c(f,[a;b]) \le \mathrm{TV}(g,[a;b]), \tag{2}$$

where $\mathrm{TV}(g,[a;b])$ is just the (total) variation of g.

Remark

The fact stated on the previous slide becomes clear when one recalls that the family of regulated functions attaing values in a Banach space coincides with the family of functions which may be uniformly approximated by finite variation functions (equivalently: by step functions) with an arbitrary accuracy.

Truncated variation - variational property

When f is a real function, then the bound (2) is attainable, i.e. there exists a function $f^c : [a;b] \to \mathbb{R}$, such that $\|f-g\|_\infty \le c/2$ and

$$\mathrm{TV}^c(f,[a;b]) = \mathrm{TV}(f^c,[a;b])$$

Thus, we have the following variational property of TV^c:

$$\mathrm{TV}^c(f,[a;b]) = \inf_{\|f-g\|_\infty \le c/2} \mathrm{TV}(g,[a;b]). \tag{3}$$

Truncated variation - variational property

When f is a real function, then the bound (2) is attainable, i.e. there exists a function $f^c : [a;b] \to \mathbb{R}$, such that $\|f - g\|_\infty \le c/2$ and

$$\mathrm{TV}^c(f, [a;b]) = \mathrm{TV}(f^c, [a;b])$$

Thus, we have the following variational property of TV^c:

$$\mathrm{TV}^c(f, [a;b]) = \inf_{\|f-g\|_\infty \le c/2} \mathrm{TV}(g, [a;b]) . \tag{3}$$

Remark

It is an open question if property (3) holds in other spaces than $\mathbb{R}$. Even for spaces E where the equality does not hold, it seems to be interesting to assess how much the left side of (3) differs from the right side.

Truncated variation - asymptotics

From definition (1) or variational property (3) (for real paths) we naturally have that

$$\lim_{c \downarrow 0} \mathrm{TV}^c(f, [a;b]) = \mathrm{TV}(f, [a;b]).$$

If $\mathrm{TV}(f, [a;b]) = +\infty$ (which is common for many important families of stochastic processes) the rate of the divergence of $\mathrm{TV}^c(f, [a;b])$ to $+\infty$ as $c \downarrow 0$ may be viewed as the measure of irregularity of the path f

$p-$variation

Naturally, in analysis there were many other notions of variations introduced. One of the most natural seems to be the $p-$variation, $p > 0$, defined as

$$\mathrm{V}^{p}(f,[a;b]) := \sup_{n} \sup_{a \le t_1 < t_2 < \ldots < t_n \le b} \sum_{i=1}^{n-1} \left| f(t_{i+1}) - f(t_i) \right|^{p}. \qquad (4)$$

For any f, if $\mathrm{V}^{p}(f,[a;b]) < +\infty$ and $q > p > 0$, then $\mathrm{V}^{q}(f,[a;b]) < +\infty$.

$p-$variation

Naturally, in analysis there were many other notions of variations introduced. One of the most natural seems to be the $p-$variation, $p > 0$, defined as

$$\mathrm{V}^{p}(f,[a;b]) := \sup_{n} \sup_{a \le t_1 < t_2 < \ldots < t_n \le b} \sum_{i=1}^{n-1} \left| f(t_{i+1}) - f(t_i) \right|^{p}. \tag{4}$$

For any f, if $\mathrm{V}^{p}(f,[a;b]) < +\infty$ and $q > p > 0$, then $\mathrm{V}^{q}(f,[a;b]) < +\infty$. This observation makes meaningful to define the *variation index*

$$Ind_{var}(f) := \inf\left\{p : \mathrm{V}^{p}(f,[a;b]) < +\infty\right\},$$

which may be also viewed as a measure of irregularity of the path - the greater $Ind_{var}(f)$, the more irregular path.

$p-$variation

Naturally, in analysis there were many other notions of variations introduced. One of the most natural seems to be the $p-$variation, $p > 0$, defined as

$$\mathrm{V}^p(f,[a;b]) := \sup_n \sup_{a \le t_1 < t_2 < \ldots < t_n \le b} \sum_{i=1}^{n-1} |f(t_{i+1}) - f(t_i)|^p . \qquad (4)$$

For any f, if $\mathrm{V}^p(f,[a;b]) < +\infty$ and $q > p > 0$, then $\mathrm{V}^q(f,[a;b]) < +\infty$. This observation makes meaningful to define the *variation index*

$$Ind_{var}(f) := \inf\{p : \mathrm{V}^p(f,[a;b]) < +\infty\},$$

which may be also viewed as a measure of irregularity of the path - the greater $Ind_{var}(f)$, the more irregular path.

Remark

If f is a step function then $Ind_{var}(f) = 0$, otherwise usually $Ind_{var}(f) \ge 1$.

The variation index vs. the asymptotics of the truncated variation

Let $\mathcal{G}([a;b])$ denote the set of real-valued, regulated functions $f:[a;b]\to\mathbb{R}$.

The variation index vs. the asymptotics of the truncated variation

Let $\mathcal{G}([a;b])$ denote the set of real-valued, regulated functions $f : [a;b] \to \mathbb{R}$.

For $p \geq 1$ let $\mathcal{V}^p([a;b])$ denote the subset of $\mathcal{G}([a;b])$ consisting of functions with finite $p-$variation.

The variation index vs. the asymptotics of the truncated variation

Let $\mathcal{G}([a;b])$ denote the set of real-valued, regulated functions $f:[a;b]\to\mathbb{R}$.

For $p\geq 1$ let $\mathcal{V}^p([a;b])$ denote the subset of $\mathcal{G}([a;b])$ consisting of functions with finite $p-$variation.

Next, for $p\geq 1$ let $\mathcal{U}^p([a;b])$ denote the subset of $\mathcal{G}([a;b])$ consisting of functions f for which

$$\limsup_{c\downarrow 0} c^{p-1}\cdot \mathrm{TV}^c(f,[a;b])<+\infty.$$

The variation index vs. the asymptotics of the truncated variation

Let $\mathcal{G}([a;b])$ denote the set of real-valued, regulated functions $f:[a;b]\to\mathbb{R}$.

For $p\geq 1$ let $\mathcal{V}^p([a;b])$ denote the subset of $\mathcal{G}([a;b])$ consisting of functions with finite $p-$variation.

Next, for $p\geq 1$ let $\mathcal{U}^p([a;b])$ denote the subset of $\mathcal{G}([a;b])$ consisting of functions f for which

$$\limsup_{c\downarrow 0} c^{p-1}\cdot \mathrm{TV}^c(f,[a;b]) < +\infty.$$

We have $\mathcal{U}^1([a;b]) = \mathcal{V}^1([a;b])$ and for any $1<p<q$ we have

$$\mathcal{V}^p([a;b]) \subsetneq \mathcal{U}^p([a;b]) \subsetneq \mathcal{V}^q([a;b]).$$

Irregularity of a typical Brownian path

If B is a standard Brownian motion, then $Ind_{var}(B) = 2$ with probability 1.

However, an old result of P. Lévy states that $\mathrm{V}^2(B, [a; b]) = +\infty$ with probability 1, thus $B \notin \mathcal{V}^2([0; t])$ with probability 1.

Irregularity of a typical Brownian path

If B is a standard Brownian motion, then $Ind_{var}(B) = 2$ with probability 1.

However, an old result of P. Lévy states that $\mathrm{V}^2(B, [a; b]) = +\infty$ with probability 1, thus $B \notin \mathcal{V}^2([0; t])$ with probability 1.

On the other hand, in [LM2013] it was proven that for any $t > 0$,

$$\lim_{c \to 0} c \cdot \mathrm{TV}^c(B, [0; t]) = t, \quad \text{with probability 1},$$

thus $B \in \mathcal{U}^2([0; t])$ with probability 1.

Irregularity of a typical Brownian path

If B is a standard Brownian motion, then $Ind_{var}(B) = 2$ with probability 1.

However, an old result of P. Lévy states that $\mathrm{V}^2(B, [a;b]) = +\infty$ with probability 1, thus $B \notin \mathcal{V}^2([0;t])$ with probability 1.

On the other hand, in [LM2013] it was proven that for any $t > 0$,

$$\lim_{c \to 0} c \cdot \mathrm{TV}^c(B, [0;t]) = t, \quad \text{with probability } 1,$$

thus $B \in \mathcal{U}^2([0;t])$ with probability 1.

Remark

In [LM2013] there was a more general fact proven: if X_t, $t \geq 0$, is a continuous semimartingale with quadratic variation $< \cdot >$ then for any $t > 0$,

$$\lim_{c \to 0} c \cdot TV^c(X, [0;t]) = < X >_t, \quad \textit{with probability 1.}$$

Irregularity of a typical Brownian path measured by $\varphi-$ variation

For a continuous, increasing function $\varphi : [0;+\infty) \to [0;+\infty)$ such that $\varphi(0) = 0$ one may define $\varphi-$variation as

$$V^{\varphi}(f,[a;b]) := \sup_{n} \sup_{a \le t_1 < t_2 < \ldots < t_n \le b} \sum_{i=1}^{n-1} \varphi(|f(t_{i+1}) - f(t_i)|). \qquad (5)$$

Irregularity of a typical Brownian path measured by $\varphi-$ variation

For a continuous, increasing function $\varphi : [0;+\infty) \to [0;+\infty)$ such that $\varphi(0) = 0$ one may define $\varphi-$variation as

$$V^{\varphi}(f,[a;b]) := \sup_{n} \sup_{a \le t_1 < t_2 < \ldots < t_n \le b} \sum_{i=1}^{n-1} \varphi\left(\left|f(t_{i+1}) - f(t_i)\right|\right). \tag{5}$$

An old result of S. J. Taylor [T1972] states that

$$\varphi_0(x) = x^2 / \ln\left(\max\left[\ln(1/x), 2\right]\right)$$

is a function with the greatest order in the neighbourhood of 0 and such that for any $t > 0$,

$$V^{\varphi_0}(B,[0;t]) < +\infty, \quad \text{with probability 1.}$$

Irregularity of a typical Brownian path measured by $\varphi-$ variation, cont.

Moreover, for any function φ such that $\lim_{x\downarrow 0} \frac{\varphi(x)}{\varphi_0(x)} = +\infty$ and $t > 0$ we have

$$\mathrm{V}^{\varphi}(B, [0; t]) = +\infty, \quad \text{with probability 1.}$$

Remark

The asymptotics of the truncated variation for functions with finite $\varphi-$variation is still to be investigated.

Truncated variation, $p-$variation and $\varphi-$variation norms of a Brownian path

The already mentioned irregularity results for the Brownian paths may be stated using $p-$variation, $\varphi-$variation and truncated variation norms, defined as

$$\|B\|_{p-var,[0;T]} := \left(\mathrm{V}^p(B,[0;T])\right)^{1/p}, \quad \text{for } p \geq 1,$$

$$\|B\|_{\varphi-var,[0;T]} := \inf\left\{\lambda > 0 : \mathrm{V}^\varphi(B/\lambda,[0;T]) \leq 1\right\},$$

and

$$\|B\|_{TV,p,[0;T]} := \left(\sup_{c>0} c^{p-1} \cdot \mathrm{TV}^c(B,[0;T])\right)^{1/p}, \quad \text{for } p \geq 1.$$

Truncated variation, $p-$variation and $\varphi-$variation norms of a Brownian path

The already mentioned irregularity results for the Brownian paths may be stated using $p-$variation, $\varphi-$variation and truncated variation norms, defined as

$$\|B\|_{p-var,[0;T]} := \left(\mathrm{V}^p(B,[0;T])\right)^{1/p}, \quad \text{for } p \geq 1,$$

$$\|B\|_{\varphi-var,[0;T]} := \inf\left\{\lambda > 0 : \mathrm{V}^{\varphi}(B/\lambda,[0;T]) \leq 1\right\},$$

and

$$\|B\|_{TV,p,[0;T]} := \left(\sup_{c>0} c^{p-1} \cdot \mathrm{TV}^c(B,[0;T])\right)^{1/p}, \quad \text{for } p \geq 1.$$

Remark

One always has

$$\|B\|_{TV,p,[0;T]} \leq \|B\|_{p-var,[0;T]}.$$

Truncated variation, $p-$variation and $\varphi-$variation norms of a Brownian path, cont.

For such defined $p-$variation, $\varphi-$variation and truncated variation norms, we have

$$\|B\|_{p,[0;T]} = \begin{cases} < +\infty & \text{if} \quad p \in (2;+\infty), \\ +\infty & \text{if} \quad p \in [1;2]; \end{cases}$$

$$\|B\|_{\varphi,[0;T]} = \begin{cases} < +\infty & \text{if} \quad \lim_{x\downarrow 0} \frac{\varphi(x)}{\varphi_0(x)} < +\infty, \\ +\infty & \text{if} \quad \lim_{x\downarrow 0} \frac{\varphi(x)}{\varphi_0(x)} = +\infty \end{cases}$$

and

$$\|B\|_{TV,p,[0;T]} = \begin{cases} < +\infty & \text{if} \quad p \in [2;+\infty), \\ +\infty & \text{if} \quad p \in [1;2). \end{cases}$$

Truncated variation vs. $p-$variation norms of a fractional Brownian path

Similarly, for a fractional Brownian B_H motion with the Hurst parameter $H \in (0;1)$ we have

$$\|B_H\|_{p,[0;T]} = \begin{cases} < +\infty & \text{if} \quad p \in (1/H; +\infty), \\ +\infty & \text{if} \quad p \in [1; 1/H] \end{cases}$$

and

$$\|B_H\|_{TV,p,[0;T]} = \begin{cases} < +\infty & \text{if} \quad p \in [1/H; +\infty), \\ +\infty & \text{if} \quad p \in [1; 1/H). \end{cases}$$

Thus, for each $T > 0$

$$B_H \in \mathcal{U}^{1/H}([0;T]) \setminus \mathcal{V}^{1/H}([0;T]), \quad \text{with probability1}.$$

Young integrals of irregular paths

Using truncated variation techniques it is possible to prove the following, stronger version of Young's result from 1936 [Young, 1936]:

Theorem

Let $f, g : [a; b] \to \mathbb{R}$ be two functions with no common points of discontinuity. If $f \in \mathcal{U}^p([a; b])$ and $g \in \mathcal{U}^q([a; b])$, where $p > 1$, $q > 1$, $p^{-1} + q^{-1} > 1$, then the Riemann Stieltjes $\int_a^b f(t)\,\mathrm{d}g(t)$ exists. Moreover, there exist a constant $C_{p,q}$, depending on p and q only, such that

$$\left|\int_a^b f(t)\,\mathrm{d}g(t) - f(a)\left[g(b) - g(a)\right]\right|$$
$$\leq C_{p,q}\|f\|_{TV,p,[a;b]}^{p-p/q}\|f\|_{osc,[a;b]}^{1+p/q-p}\|g\|_{TV,q,[a;b]},$$

where $\|f\|_{osc,[a;b]} := \sup_{a\leq s<t\leq b}|f(t) - f(s)|$.

How to calculate the truncated variation?

There exists an algorithm based on drawdown and drawup times. Let us define

$$T_U^c f = \inf\left\{ t \in (a; b] : f(t) - \inf_{a \le s \le t} f(s) > c \right\}$$

$$T_D^c f = \inf\left\{ t \in (a; b] : \sup_{a \le s \le t} f(s) - f(t) > c \right\}$$

How to calculate the truncated variation?

There exists an algorithm based on drawdown and drawup times. Let us define

$$T_U^c f = \inf\left\{ t \in (a;b] : f(t) - \inf_{a \le s \le t} f(s) > c \right\}$$

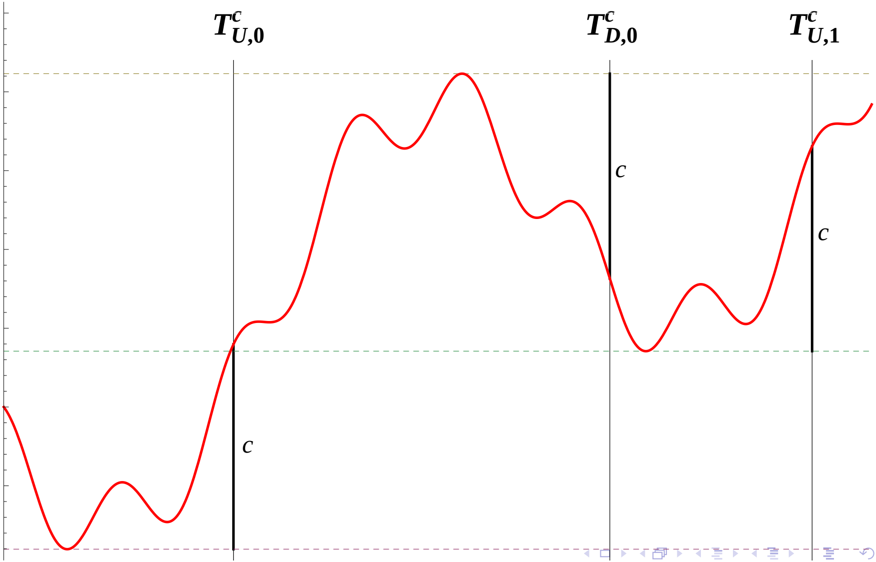
$$T_D^c f = \inf\left\{ t \in (a;b] : \sup_{a \le s \le t} f(s) - f(t) > c \right\}$$

If $\min\left\{ T_U^c f, T_D^c f \right\} = +\infty$, then $\mathrm{TV}^c(f, [a;b]) = 0$. If not, assuming that $T_U^c f < T_D^c f$ we define $T_{D,-1}^c f = a$ and then for $k = 0, 1, 2, \ldots$

$$T_{U,k}^c f = \inf\left\{ t \in (T_{D,k-1}^c f; b] : f(t) - \inf_{T_{D,k-1}^c f \le s \le t} f(s) > c \right\},$$

$$T_{D,k}^c f = \inf\left\{ t \in (T_{U,k}^c f; b] : \sup_{T_{U,k}^c f \le s \le t} f(s) - f(t) > c \right\}.$$

Times $T^c_{U,k}$, $T^c_{D,k}$, $k = 0, 1, \ldots$



Calculation of the truncated variation, cont.

Now, to calculate the truncated variation we define

$$m_k = \inf_{s\in[T_{D,k-1};T_{U,k}]} f(s), \quad M_k = \inf_{s\in[T_{U,k};T_{D,k}]} f(s),$$

and for t such that $t \in [T_{U,k}; T_{D,k}]$ we have

$$\begin{aligned} TV^c(f,[a;t]) = & \sum_{i=0}^{k-1} (M_i - m_i - c) + \sum_{i=0}^{k-1} (M_i - m_{i+1} - c) \\ & + \sup_{s\in[T^c_{U,k}f,t]} f(s) - m_k - c. \end{aligned} \tag{6}$$

Similarly, for t such that $t \in [T_{D,k}; T_{U,k+1}]$ we have

$$\begin{aligned} TV^c(f,[a;t]) = & \sum_{i=0}^{k} (M_i - m_i - c) + \sum_{i=0}^{k-1} (M_i - m_{i+1} - c) \\ & + M_k - \inf_{s\in[T^c_{D,k}f,t]} f(s) - c. \end{aligned} \tag{7}$$

Stopping times $T^c_{U,k}$, $T^c_{D,k}$, $k = 0, 1, \ldots$

If X_t, $t \geq 0$, is a stochastic process with càdlàg (or even regulated!) trajectories, adapted to the filtration $\mathcal{F}_t$, $t \geq 0$, then for each trajectory $f = X(\omega)$ the just defined times $T^c_{U,k}$, $T^c_{D,k}$, $k = 0, 1, \ldots$ are stopping times such that

$$\lim_{k \to +\infty} T^c_{U,k} = \lim_{k \to +\infty} T^c_{D,k} = +\infty, \quad \text{with probability 1.}$$

Stopping times $T^c_{U,k}$, $T^c_{D,k}$, $k = 0, 1, \ldots$

If X_t, $t \geq 0$, is a stochastic process with càdlàg (or even regulated!) trajectories, adapted to the filtration $\mathcal{F}_t$, $t \geq 0$, then for each trajectory $f = X(\omega)$ the just defined times $T^c_{U,k}$, $T^c_{D,k}$, $k = 0, 1, \ldots$ are stopping times such that

$$\lim_{k \to +\infty} T^c_{U,k} = \lim_{k \to +\infty} T^c_{D,k} = +\infty, \quad \text{with probability 1.}$$

To see this, note that for any $t > 0$ there exists such $K < +\infty$ that $T^c_{U,K} > t$ and $T^c_{D,K} > t$. Otherwise we would obtain two infinite sequences $(s_k)_{k=1}^{\infty}, (S_k)_{k=1}^{\infty}$ such that $0 \leq s_1 < S_1 < s_2 < S_2 < ... < t$ and $f(S_k) - f(s_k) \geq \frac{1}{2}c$. But this is a contradiction, since f is a regulated function and $(f(s_k))_{k=1}^{\infty}, (f(S_k))_{k=1}^{\infty}$ have a common limit.

Calculation of the truncated variation of a process with continuous trajectories

When X_t, $t \geq 0$, is a stochastic process with continuous trajectories then for any $t > 0$, $\mathrm{TV}^c(X, [0; t])$ may be calculated with the stochastic sampling scheme based on the stopping times $T^c_{U,k}$, $T^c_{D,k}$, $k = 0, 1, \ldots$. Indeed, for continuous $f = X(\omega)$ we have $X_{T^c_{U,k}}(\omega) = f(T^c_{U,k}) = m_k + c$ and $X_{T^c_{D,k}}(\omega) = f(T^c_{D,k}) = M_k - c$, hence from formulas (6) and (7) we have

$$\mathrm{TV}^c\left(X, [0; T^c_{U,k}]\right) = \sum_{i=0}^{k-1} \left(X_{T^c_{D,i}} - X_{T^c_{U,i}} + c \right) + \sum_{i=0}^{k-1} \left(X_{T^c_{D,i}} - X_{T^c_{U,i+1}} + c \right).$$

Similarly,

$$\mathrm{TV}^c\left(X, [0; T^c_{D,k}]\right) = \sum_{i=0}^{k} \left(X_{T^c_{D,i}} - X_{T^c_{U,i}} + c \right) + \sum_{i=0}^{k-1} \left(X_{T^c_{D,i}} - X_{T^c_{U,i+1}} + c \right).$$

Realized volatility estimation

If X_t, $t \geq 0$, is a continuous semimartingale then the (mentioned already) relation

$$c \cdot \mathrm{TV}^c(X, [0; t]) \to < X >_t, \quad \text{with probability 1} \tag{8}$$

(in the uniform convergence topology on compacts) and the stated algorithm provides us with a tool for the realized volatility estimation.

Realized volatility estimation

If X_t, $t \geq 0$, is a continuous semimartingale then the (mentioned already) relation

$$c \cdot \mathrm{TV}^c(X, [0; t]) \to < X >_t, \quad \text{with probability 1} \tag{8}$$

(in the uniform convergence topology on compacts) and the stated algorithm provides us with a tool for the realized volatility estimation.

Remark

The rate of convergence in (8) is better than the rate of the usual algorithms based on the calendar time or business time sampling (this was proved at least for some class of diffusions), and is of the same order as the order of tick time sampling schemes, which are special case of stochastic sampling schemes for realized volatility estimation introduced by Masaaki Fukasawa [F2010].

Rate of convergence in (8)

Let X_t, $t \geq 0$ be the strong solution of the SDE $\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t$, where B is a standard Brownian motion and μ, $\sigma > 0$ satisfy "usual" conditions guaranteeing the existence uniqueness of the strong solution of SDE. In [LM2013] the following result was proven

$$\frac{1}{c}\left(c \cdot \mathrm{TV}^c(X,[0;t]) - < X >_t\right) \Rightarrow W_{<X>_t/3},$$

where W is another standard Brownian motion, independent from B and the convergence $\Rightarrow$ is understood as the stable convergence in the uniform convergence topology on compacts.

Stochastic sampling schemes for realized volatility estimation

Masaaki Fukasawa considered general stochastic sampling schemes and obtained similar rates of convergence for tick sampling.

He considered a continuous semimartingale $X = A + M$, where M_t, $t \geq 0$, is a continuous local martingale adapted to some filtration $\mathcal{F}_t$, $t \geq 0$, and A_t, $t \geq 0$, is a finite variation process such that $A_t = \int_0^t \psi_s \mathrm{d} < M >_s$, where ψ_s, $t \geq 0$, is locally bounded, left continuous process adapted to $\mathcal{F}_t$ and a sequence of $\mathcal{F}_t-$stopping times $0 = \tau_0 < \tau_1 < \ldots$ such that for every $T > 0$

$$N_\tau(t) := \max\{k \geq 0 : \tau_k > t\} < +\infty, \quad \text{with probability 1.}$$

Stochastic sampling schemes for realized volatility estimation, cont.

The statistics

$$RV_{\tau}(t) := \sum_{k=0}^{N_{\tau}(t)-1} \left(X_{\tau_{k+1}} - X_{\tau_k}\right)^2$$

is an estimator of $< X >_t$ and the rate of convergence of this statistics depends on the sequence $\tau = \{\tau_0 < \tau_1 < \ldots\}$.

The best rates of convergence one obtains taking e.g. tick sampling, i.e. $\tau_0^c = 0$ and for $k = 0, 1, \ldots$

$$\tau_{k+1}^c = \inf\left\{t > t_k : \left|X_t - X_{\tau_k^c}\right| > c\right\},$$

and we have stable convergence to a non-trivial limit of the processes

$$\frac{1}{c}\left(RV_{\tau^c}(t) - < X >_t\right).$$

Stochastic sampling schemes for realized volatility estimation, general remarks

- With the tick time sampling (based on stopping times τ_k^c) (or the truncated variation sampling based on stopping times $T_{U,k}^c$, $T_{D,k}^c$,) we need to control the process very precisely (**no unobserved big fluctuations between consecutive samplings**) like in the case of the calendar time sampling $\tau_k = k/n$.
- As a reward for this we obtain a better mode of the first order convergence (almost sure convergence instead of the convergence in probability) and a better asymptotics of the second order convergence.
- This is no problem as long we assume that we are able to observe **rounded** values of the process X at **any** time.
- The tick time sampling or the truncated variation sampling are robust to the price rounding (market microstructure noise).

Variational properties of the tick sampling scheme

The tick time sampling scheme has also interesting property regarding the total variation of the càdlàg process $\hat{X}^c$ defined as

$$\hat{X}_t^c = X_{\tau_k^c}, \quad \text{where} \quad k = \max\{i = 0, 1, 2, \ldots : \tau_i^c \le t\}.$$

This process approximates the process X with accuracy c and we have

$$\mathrm{TV}\left(\hat{X}^c, [0; t]\right) = c \cdot N_\tau(t)$$

and

$$RV_{\tau^c}(t) := c^2 \cdot N_{\tau^c}(t) = c \cdot \mathrm{TV}\left(\hat{X}^c, [0; t]\right).$$

Thus, recalling the convergence results for $RV_{\tau^c}(t)$ we get that $\mathrm{TV}\left(\hat{X}^c, [0; t]\right)$ has the same magnitude as $\mathrm{TV}^c(X, [0; t])$ and it approximates uniformly X with accuracy c (which differs from the optimal accuracy for a process with the total vatiation $\mathrm{TV}^c(X, [0; t])$ by factor 2).

The truncated variation and segment crossings

The truncated variation appears to be esspecially relevant for the investigation of the numbers of segment crossings. First, let us recall the Banch Indicatrix theorem. If $f : [a, b] \to \mathbb{R}$ is a continuous function then

$$\mathrm{TV}(f, [a; b]) = \int_{\mathbb{R}} N^y (f, [a; b]) \, \mathrm{d}y,$$

where

$$N^y (f, [a; b]) := \mathrm{card} \{x \in [a; b] : f(x) = y\}$$

is called the Banach indicatrix. A generalisation of this result for the case of regulated f is possible.
Unfortunately, when $\mathrm{TV}(f, [a; b]) = +\infty$ this result seems to be useless.

The truncated variation and segment crossings, cont.

However, when instead of considering the level crossings, one considers *segment crossings* and instead of considering the total variation one considers *the truncated variation* then one gets the following result.
For any regulated $f : [a; b] \to \mathbb{R}$ and $c > 0$,

$$\mathrm{TV}^c(f, [a; b]) = \int_{\mathbb{R}} n_c^y(f, [a; b])\, \mathrm{d}y. \tag{9}$$

Here,

$$n_c^y(f, [a; b]) = \text{number of times that } f \text{ crosses the segment } [y; y + c]. \tag{10}$$

Segment crossings - precise definition

To be more precise, we define

$$n_c^y(f,[a,b]) = d_c^y(f,[a,b]) + u_c^y(f,[a,b]).$$

$d_c^y(f,[a,b])$ = number of downcrossings the segment $[y; y+c]$..

$\sigma_0 = a$, and for $k = 0, 1, \ldots$

$$\nu_k = \inf\{t > \sigma_k : f(t) > y+c\} \quad \sigma_{k+1} = \inf\{t > \nu_k : f(t) < y\}.$$

Now we define

$$d_c^y(f,[a,b]) = \max\{k : \sigma_k \le b\}.$$

Segment crossings - precise definition

To be more precise, we define

$$n_c^y(f,[a,b]) = d_c^y(f,[a,b]) + u_c^y(f,[a,b]).$$

$d_c^y(f,[a,b]) =$ number of downcrossings the segment $[y; y+c]$..

$\sigma_0 = a$, and for $k = 0, 1, \ldots$

$$\nu_k = \inf\{t > \sigma_k : f(t) > y + c\} \quad \sigma_{k+1} = \inf\{t > \nu_k : f(t) < y\}.$$

Now we define

$$d_c^y(f,[a,b]) = \max\{k : \sigma_k \leq b\}.$$

$d_c^y(f,[a,b]) =$ number of downcrossings the segment $[y; y+c]$.

$\tilde{\sigma}_0 = a$, and for $k = 0, 1, \ldots$

$$\tilde{\nu}_k = \inf\{t > \tilde{\sigma}_k : f(t) < y\} \quad \tilde{\sigma}_{k+1} = \inf\{t > \tilde{\nu}_k : f(t) > y + c\}.$$

$$u_c^y(f,[a,b]) = \max\{k : \tilde{\sigma}_k \leq b\}.$$

Limit theorems for segment crossings

Relation (9), linking the truncated variation with the numbers of segment crossings, allows to obtain, for relatively broad spectrum of stochastic processes, limit theorems for the numbers of segment crossings of these processes.

For example, for a continuous semimartingale X_t, $t \geq 0$, using the already mentioned convergence

$$c \cdot \mathrm{TV}^c(X, [0; t]) \to < X >_t, \quad \text{with probability 1}$$

one obtains the convergence

$$\int_{\mathbb{R}} c \cdot n_c^y(X, [0; t]) \, \mathrm{d}y \to < X >_t, \quad \text{with probability 1.}$$

It is also possible to make some of the levels more important than others, by introducing continuous density $g : \mathbb{R} \to \mathbb{R}$, and obtain

$$\int_{\mathbb{R}} c \cdot n_c^y(X, [0; t]) \, g(y) \mathrm{d}y \to \int_0^t g(X_s) \, \mathrm{d} < X >_s, \quad \text{with probability 1.} \tag{11}$$

Limit theorems for segment crossings, diffusions

Similalry, from the already mentioned convergence

$$\left(\mathrm{TV}^c(X,[0;t]) - \frac{1}{c} < X >_t\right) \Rightarrow W_{<X>_t/3},$$

for X_t, $t \geq 0$, being the strong solution of the SDE $\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t$, where B is a standard Brownian motion and μ, $\sigma > 0$ satisfy "usual" conditions guaranteeing the existence uniqueness of the strong solution of SDE, and any continuous function $g : \mathbb{R} \to \mathbb{R}$, the following convergence of the integrated numbers of segment crossings was obtained ([LG2014]):

$$\left(\int_{\mathbb{R}} n_c^y(X,[0;t])\,g(y)\mathrm{d}y - \frac{1}{c}\int_0^t g(X_s)\,\mathrm{d} < X >_s\right) \Rightarrow \int_0^t g(X_s)\,W_{\frac{<X>_t}{3}},$$

where B and W are two independent standard Brownian motions.

Limit theorems for segment down-, up- crossings, diffusions

Together with the convergence of segment crossings, one has (obvious) first order convergence of segment down- and up- crossings (since $u_c^y(X, [0;t]) - d_c^y(X, [0;t]) \in \{-1, 0, 1\}$):

$$\int_{\mathbb{R}} c \cdot u_c^y(X, [0;t])\, g(y) \mathrm{d}y \to \frac{1}{2} \int_0^t g(X_s) \,\mathrm{d} < X >_s, \quad \text{with probability 1.}$$

But in the case of the second order convergence an interesting correction term (the Stratonovich integral) appears:

$$\int_{\mathbb{R}} u_c^y(X, [0;t])\, g(y) \mathrm{d}y - \frac{1}{2 \cdot c} \int_0^t g(X_s) \,\mathrm{d} < X >_s$$
$$\Rightarrow \frac{1}{2} \int_0^t g(X_s)\, W_{\frac{<X>_t}{3}} + \frac{1}{2} \int_0^t g(X_s) \circ \mathrm{d}X_s,$$

where B and W are two independent standard Brownian motions.

Segment crossings and local times of diffusions

Let $L_t^y(X)$ be the local time of X at $y \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Recalling the occupation times formula

$$\int_0^t g(X_s) \mathrm{d} < X >_s = \int_{\mathbb{R}} g(y) L_t^y(X) \mathrm{d}y.$$

we get

$$\int_{\mathbb{R}} \left[n_c^y(X, [0; t]) - \frac{1}{c} L_t^y(X) \right] g(y) \mathrm{d}y \Rightarrow \int_0^t g(X_s) W_{\frac{<X>_t}{3}}.$$

Segment crossings and local times of diffusions

Let $L_t^y(X)$ be the local time of X at $y \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Recalling the occupation times formula

$$\int_0^t g(X_s)\,\mathrm{d} < X >_s = \int_{\mathbb{R}} g(y)\, L_t^y(X)\,\mathrm{d}y.$$

we get

$$\int_{\mathbb{R}} \left[n_c^y(X,[0;t]) - \frac{1}{c} L_t^y(X) \right] g(y)\,\mathrm{d}y \Rightarrow \int_0^t g(X_s)\, W_{\frac{<X>_t}{3}}.$$

On the other hand, we have the following convergence ([Kasahara, 1980] for a standard Brownian motion, [LG2014, Theorem 4.5] for diffusions)

$$\sqrt{c} \left[n_c^y(X,[0;t]) - \frac{1}{c} L_t^y(X) \right] \Rightarrow W_{L_t^y(X)}.$$

Segment crossings and local times of diffusions

Let $L_t^y(X)$ be the local time of X at $y \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Recalling the occupation times formula

$$\int_0^t g(X_s)\,\mathrm{d} < X >_s = \int_{\mathbb{R}} g(y)\, L_t^y(X)\,\mathrm{d}y.$$

we get

$$\int_{\mathbb{R}} \left[n_c^y(X, [0;t]) - \frac{1}{c} L_t^y(X) \right] g(y)\,\mathrm{d}y \Rightarrow \int_0^t g(X_s)\, W_{\frac{<X>_t}{3}}.$$

On the other hand, we have the following convergence ([Kasahara, 1980] for a standard Brownian motion, [LG2014, Theorem 4.5] for diffusions)

$$\sqrt{c} \left[n_c^y(X, [0;t]) - \frac{1}{c} L_t^y(X) \right] \Rightarrow W_{L_t^y(X)}.$$

Thus we see that integrating differences $n_c^y(X, [0;\cdot]) - \frac{1}{c} L^y(X)$ we get much faster convergence (multiplication by $\sqrt{c}$ is no needed).

"Meta-theorem", almost sure convergence

For a process $X_t, t \geq 0$, let us denote $\mathrm{TV}^c(X, \cdot) := \mathrm{TV}^c(X, [0; \cdot])$ and $n_c^a(X, \cdot) := n_c^a(X, [0; \cdot])$.

"Meta-theorem", almost sure convergence

For a process $X_t, t \geq 0$, let us denote $\mathrm{TV}^c(X,\cdot) := \mathrm{TV}^c(X,[0;\cdot])$ and $n_c^a(X,\cdot) := n_c^a(X,[0;\cdot])$.

Theorem (M1)

Let $X_t, t \geq 0$, be a càdlàg process and assume that there exists an increasing function $\varphi : (0;+\infty) \to (0;+\infty)$, such that $\lim_{c\to 0+} \varphi(c) = 0$, and a càdlàg process $\zeta_t, t \geq 0$, with locally finite variation, such that the following convergence holds

$$\varphi(c)\, TV^c(X,\cdot) \to \zeta.$$

Then for any continuous function $f : \mathbb{R} \to \mathbb{R}$ we have the following convergence

$$\varphi(c) \int_{\mathbb{R}} n_c^a(X,\cdot) f(a)\, \mathrm{d}a \to \int_0^{\cdot} f(X_{t-})\, \mathrm{d}\zeta_t.$$

"Meta-theorem", stable convergence

Theorem (M2)

Let $X_t, t \geq 0$, *be a càdlàg process and assume that there exists an increasing function* $\varphi : (0; +\infty) \to (0; +\infty)$, *with* $\lim_{c \to 0+} \varphi(c) = 0$, *and a càdlàg process* ζ_t, $t \geq 0$, *with locally finite variation, such that the following convergence holds*

$$\varphi(c)\, TV^c(X, \cdot) \Rightarrow \zeta$$

then for any continuous function $f : \mathbb{R} \to \mathbb{R}$ *we have the following convergence*

$$\varphi(c) \int_{\mathbb{R}} n_c^a(X, \cdot) f(a) \mathrm{d}a \Rightarrow \int_0^{\cdot} f(X_{t-})\, \mathrm{d}\zeta_t.$$

Self-similar processes

Theorem (SsP)

Let $X_t, t \geq 0$, be a càdlàg process such that it has stationary increments. Next, assume that X is a self-similar proces, that is, there exists $\beta \in (0;1)$ such that

$$\left\{A^{-\beta} X_{At}, t \geq 0\right\} =^d \{X_t, t \geq 0\}$$

for any $A > 0$. Additionally we assume that for some $c > 0$, $\mathbb{E} TV^c(X, [0,1]) < +\infty$ and that the tail σ-field is trivial. Then

$$c^{1/\beta - 1} TV^c(X, \cdot) \to C \cdot Id.$$

Lévy processes

Theorem (LP1)

Let $X_t, t \geq 0$, *be a Lévy process which has infinite total variation and is no monotonic on any non-degenerate interval. Moreover, assume that* $\mathbb{E}\sup_{0\leq t<T_D^{c_0}X} X_t < +\infty$ *for some* $c_0 > 0$, *where we define*

- $T_D^c X := \inf\left\{t \geq 0 : \sup_{0\leq s\leq t} X_s - X_t > c\right\}, \quad \theta_U^c := \mathbb{E} T_D^c X$

Lévy processes

Theorem (LP1)

Let $X_t, t \geq 0$, be a Lévy process which has infinite total variation and is no monotonic on any non-degenerate interval. Moreover, assume that $\mathbb{E}\sup_{0\leq t<T_D^{c_0}X} X_t < +\infty$ for some $c_0 > 0$, where we define

- $T_D^c X := \inf\left\{t \geq 0 : \sup_{0\leq s\leq t} X_s - X_t > c\right\}, \quad \theta_U^c := \mathbb{E} T_D^c X$
- $\xi_U^c := \sup_{0\leq s<t<T_D^c X} (X_t - X_s - c)_+, \quad \eta_U^c := \mathbb{E}\xi_U^c.$

Lévy processes

Theorem (LP1)

Let $X_t, t \geq 0$, be a Lévy process which has infinite total variation and is no monotonic on any non-degenerate interval. Moreover, assume that $\mathbb{E} \sup_{0 \leq t < T_D^{c_0} X} X_t < +\infty$ for some $c_0 > 0$, where we define

- $T_D^c X := \inf \left\{ t \geq 0 : \sup_{0 \leq s \leq t} X_s - X_t > c \right\}, \quad \theta_U^c := \mathbb{E} T_D^c X$
- $\xi_U^c := \sup_{0 \leq s < t < T_D^c X} (X_t - X_s - c)_+, \quad \eta_U^c := \mathbb{E} \xi_U^c.$

If for

$$\chi_U(c) := \theta_U^c / \eta_U^c$$

and for any $u > 0$, $\mathbb{P}\left(\xi_U^c \leq u/\chi_U(c)\right)/\theta_U^c \to 1$ as $c \to 0+$, then we have the following convergence

$$\chi_U(c)\, TV^c(X, \cdot) \Rightarrow 2 \cdot Id.$$

Lévy processes, cont.

Theorem (LP2)

Let $X_t, t \geq 0$, *be a Lévy process which has infinite total variation and is no monotonic on any non-degenerate interval. Moreover, assume that* $\mathbb{E} \sup_{0 \leq t < T_D^{c_0} X} X_t < +\infty$ *for some* $c_0 > 0$, *where we define*

- $T_U^c X := \inf \{ t \geq 0 : X_t - \inf_{0 \leq s \leq t} X_s > c \}, \quad \theta_D^c := \mathbb{E} T_D^c X$

Lévy processes, cont.

Theorem (LP2)

Let $X_t, t \geq 0$, be a Lévy process which has infinite total variation and is no monotonic on any non-degenerate interval. Moreover, assume that $\mathbb{E}\sup_{0 \leq t < T_D^{c_0} X} X_t < +\infty$ for some $c_0 > 0$, where we define

- $T_U^c X := \inf\left\{t \geq 0 : X_t - \inf_{0 \leq s \leq t} X_s > c\right\}, \quad \theta_D^c := \mathbb{E} T_D^c X$
- $\xi_D^c := \sup_{0 \leq s < t < T_U^c X} \left(X_s - X_t - c\right)_+, \quad \eta_D^c := \mathbb{E}\xi_D^c.$

Lévy processes, cont.

Theorem (LP2)

Let $X_t, t \geq 0$, be a Lévy process which has infinite total variation and is no monotonic on any non-degenerate interval. Moreover, assume that $\mathbb{E}\sup_{0\leq t<T_D^{c_0}X} X_t < +\infty$ for some $c_0 > 0$, where we define

- $T_U^c X := \inf\{t \geq 0 : X_t - \inf_{0\leq s\leq t} X_s > c\}, \quad \theta_D^c := \mathbb{E}T_D^c X$
- $\xi_D^c := \sup_{0\leq s<t<T_U^c X}(X_s - X_t - c)_+, \quad \eta_D^c := \mathbb{E}\xi_D^c.$

If for

$$\chi_D(c) := \theta_D^c/\eta_D^c$$

and for any $u > 0$, $\mathbb{P}(\xi_D^c \leq u/\chi_D(c))/\theta_D^c \to 1$ as $c \to 0+$, then we have the following convergence

$$\chi_D(c)\, TV^c(X,\cdot) \Rightarrow 2 \cdot Id.$$

Consequence of Theorem M1 and Theorem SsP

Corollary (S)

Let $X_t, t \geq 0$, *be a self-similar càdlàg process as in Theorem SsP and* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function, then*

$$c^{1/\beta - 1} \int_{\mathbb{R}} n_c^a(X, \cdot) f(a) \, \mathrm{d}a \to C \int_0^{\cdot} f(X_{t-}) \, \mathrm{d}t,$$

where the constant C *is the same as in Theorem SsP.*

Consequence of Theorem M2 and Theorems LP1, LP2

Corollary (L)

Let $X_t, t \geq 0$, be a Lévy process as in Theorem L1. If the assumptions of Theorem L1 are satisfied then

$$\chi_U(c) \int_{\mathbb{R}} n_c^a(X, \cdot) f(a) \, \mathrm{d}a \Rightarrow 2 \int_0^{\cdot} f(X_{t-}) \, \mathrm{d}t.$$

An analogous convergence, namely

$$\chi_D(c) \int_{\mathbb{R}} n_c^a(X, \cdot) f(a) \, \mathrm{d}a \Rightarrow 2 \int_0^{\cdot} f(X_{t-}) \, \mathrm{d}t,$$

holds when the assumptions of Theorem L2 are satisfied.

More specific case - spectrally assymetric processes with "almost" α-stable jumps

Theorem

Let X_t, $t \geq 0$, be a Lévy process without Brownian component, with the Lévy measure Π such that

$$\Pi(\mathrm{d}x) = \frac{L(x)}{(-x)^{1+\alpha}} 1_{x<0} \mathrm{d}x$$

for $\alpha \in (1;2)$ and some Borel-measurable function $L : (-\infty;0) \to (0;+\infty)$, slowly varying at 0. Then $\chi_D(c) \sim \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \frac{c^{\alpha-1}}{L(-c)}$ and

$$\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \frac{c^{\alpha-1}}{L(-c)} TV^c(X,\cdot) \Rightarrow 2 \cdot Id.$$

A remark on local times

The most common definition of the local times $L = L_t^a$, $a \in \mathbb{R}$, $t \geq 0$, of a given process X_t, $t \geq 0$, is as the Radon-Nikodym derivative of the occupation measure of X with respect to the Lebesgue measure in $\mathbb{R}$; for every Borel measurable function $f : \mathbb{R} \to \mathbb{R}$ and $t > 0$,

$$\int_0^t f(X_s)\,\mathrm{d}s = \int_{\mathbb{R}} f(a)\,L_t^a \mathrm{d}a.$$

A remark on local times

The most common definition of the local times $L = L_t^a$, $a \in \mathbb{R}$, $t \geq 0$, of a given process X_t, $t \geq 0$, is as the Radon-Nikodym derivative of the occupation measure of X with respect to the Lebesgue measure in $\mathbb{R}$; for every Borel measurable function $f : \mathbb{R} \to \mathbb{R}$ and $t > 0$,

$$\int_0^t f(X_s)\,\mathrm{d}s = \int_{\mathbb{R}} f(a)\,L_t^a \mathrm{d}a.$$

Notice that for any càdlàg process X_t, $t \geq 0$, and continuous f

$$\int_0^{\cdot} f(X_{t-})\,\mathrm{d}t = \int_0^{\cdot} f(X_t)\,\mathrm{d}t.$$

Thus, in view of Corollary S and Corollary L we have that the natural candidates for the local times L_t^a for a self-similar or a Lévy process X are the limits (if they exist)

$$C^{-1} c^{1/\beta - 1} n_c^a(X,t), \quad \frac{1}{2}\chi_U(c)\, n_c^a(X,t) \quad \text{or} \quad \frac{1}{2}\chi_D(c)\, n_c^a(X,t).$$

Second order convergence of the truncated variation of strictly α-stable processes

Let X_t, $t \geq 0$, be strictly α-stable process with the characteristic exponent

$$\Psi_X(\theta) = C_0 |\theta|^\alpha \left(1 - i\gamma \tan\frac{\pi\alpha}{2} \operatorname{sgn}\theta\right), \tag{12}$$

where $\alpha \in (1;2)$ is the index, $C_0 > 0$ is the scale parameter and $\gamma \in [-1;1]$ is the skewness parameter. Define

$$A := \lim_{N \to +\infty} \mathbb{E}\left(\mathrm{TV}^1(X,[0;N+1]) - \mathrm{TV}^1(X,[0;N])\right).$$

(it is possible to prove that this limit exists) and

$$T_t^c := \mathrm{TV}^c(X,[0,t]) - c^{1-\alpha} A \cdot t.$$

Second order convergence of the truncated variation of strictly α-stable processes, cont.

Theorem (SecondOrder)

Let $\alpha \in (1;2)$ and X_t, $t \geq 0$, be strictly α-stable process with the characteristic exponent given by formula (12), then

$$T^c \Longrightarrow L^1 + L^2,$$

where L^1 and L^2 are two independent, spectrally positive processes such that $X = L^1 - L^2$ and $L^1 + L^2$ is strictly α-stable, spectrally positive process with the characteristic exponent given by formula

$$\Psi_{L^1+L^2}(\theta) = C_0|\theta|^\alpha \left(1 - i\tan\frac{\pi\alpha}{2} sgn\theta\right).$$

Second order convergence of the truncated variation of strictly 1-stable processes

Let X_t, $t \geq 0$, be strictly 1-stable process with the characteristic exponent

$$\Psi_X(\theta) = C_0|\theta| + i\eta\theta, \tag{13}$$

with the scale parameter $C_0 > 0$ and the drift $\eta \in \mathbb{R}$. (The characterisitc exponent of a strictly 1-stable process is necessarily of this form.) Let us set

$$B = \lim_{N \to +\infty} \mathbb{E}\left(\mathrm{TV}^1(X,[0;N+1]) - \mathrm{TV}^1(X,[0;N]) - \mathrm{TV}(Y,[0;N])\right),$$

where $Y = \sum_{N<s\leq N+1} |X_s - X_{s-}| 1_{|X_s - X_{s-}|\geq 1}$ and

$$T_t^c := \mathrm{TV}^c(X,[0,t]) - \frac{2}{\pi} C_0 \log c^{-1} \cdot t - B \cdot t.$$

Second order convergence of the truncated variation of strictly 1-stable processes, cont.

Theorem (SecondOrder1)

Let X_t, $t \geq 0$, be strictly 1-stable process with the characteristic exponent given by formula (13), then

$$T^c \Longrightarrow M^1 + M^2,$$

where M^1, M^2 are two independent, spectrally positive processes such that $X = M^1 - M^2$ and $M^1 + M^2$ is 1-stable process with the characteristic exponent given by formula

$$\Psi_{M^1+M^2}(\theta) = C_0|\theta|\left(1 + i\frac{2}{\pi}sgn(\theta)\log|\theta|\right) - i\frac{2(1-\mathbb{C})}{\pi}C_0\theta,$$

where $\mathbb{C} = \Gamma'(1) \approx 0.577$ is the Euler-Mascheroni constant.

Second order convergences for numbers of segment crossings

The immediate consequences of Theorem SecondOrder, SecondOrder1 and equality (10) is the second order convergence for the integrated numbers of segment crossings:

- if X is strictly α-stable ($\alpha \in (1;2)$):

$$\int_{\mathbb{R}} n_c^y(X,t)\,\mathrm{d}y - c^{1-\alpha} A \cdot t \Longrightarrow L^1 + L^2$$

- and if X is strictly 1-stable:

$$\int_{\mathbb{R}} n_c^y(X,t)\,\mathrm{d}y - \frac{2}{\pi} C_0 \log c^{-1} \cdot t - B \cdot t \Longrightarrow M^1 + M^2.$$

Open questions

Naturally, the next step would be the investigation of the second order convergence for the integrated numbers of segment crossings with respect to the measure $f(y)\mathrm{d}y$:

$$\int_{\mathbb{R}} n_c^y(X,t)\, f(y)\mathrm{d}y.$$

Why it may be interesting?

Open questions

Naturally, the next step would be the investigation of the second order convergence for the integrated numbers of segment crossings with respect to the measure $f(y)\mathrm{d}y$:

$$\int_{\mathbb{R}} n_c^y(X,t)\, f(y)\mathrm{d}y.$$

Why it may be interesting?

1. The integral $\int_{\mathbb{R}} n_c^y(X,t)\, f(y)\mathrm{d}y$ may reveal much stronger concentration than $n_c^y(X,t)$ for given $y \in \mathbb{R}$.
2. Investigation together with $\int_{\mathbb{R}} n_c^y(X,t)\, f(y)\mathrm{d}y$ integrals of the form $\int_{\mathbb{R}} d_c^y(X,t)\, f(y)\mathrm{d}y$, where $d_c^y(X,t)$ is the number of downcrossings of X from above the level $y+c$ to the level y till time t, may reveal interesting correction terms.

Some references

[F2010] M. Fukasawa, Central limit theorem for the realized volatility based on tick time sampling. *Fin. Stoch.* 14:209–233, 2010.

[Kasahara, 1980] Y. Kasahara. On Lévy's downcrossing theorem. *Proc. Japan Acad. Ser. A Math. Sci.*, 56:455–458, 1980.

[LG2014] R. M. Łochowski and R. Ghomrasni. Integral and local limit theorems for numbers of level crossings for diffusions and the Skorohod problem. *Electron. J. Probab.* 19(10):1–33, 2014.

[LM2013] R. M. Łochowski and P. Miłoś. On truncated variation, upward truncated variation and downward truncated variation for diffusions. *Stochastic Process. Appl.* 123:446-474.

[LM2014] R. M. Łochowski and P. Miłoś. Limit theorems for the truncated variation and for numbers of interval crossings of Lévy and self-similar processes. *preprint*, available from the web page `www.akson.sgh.waw.pl\~rlocho`

Some references, cont.

[L2014] R. M. Łochowski. On the existence of the Riemann-stieltjes integral. *Preprint arXiv:1403.5413*, 2014.

[T1972] S. J. Taylor. Exact asymptotic estimates of Brownian path variation. *Duke Math. J.*, 39(2):219–241, 1972.

[Young, 1936] L. C. Young. An inequality of the Hölder type, connected with Stieltjes integration. *Acta Math.*, 67(1):251–282, 1936.

Thank you!