Pathwise quadratic variation and local times of deterministic càdlàg paths - linking several approaches

Rafał M. Łochowski
Warsaw School of Economics
joint work with David J. Prömel, Jan Obłój (University of Oxford) and Pietro Siorpaes (Imperial College)

AIMS Journal Club Talk, Muizenberg 2019

Brief history of pathwise stochastic calculus

- In his seminal paper [F8̈1] Föllmer proved that properly defined quadratic variation of a deterministic càdlàg path (as the limit of discrete quadratic variations along some sequence of partitions) leads to the same formula as **Itô's formula** for càdlàg semimartingales.
- Bichteler ([Bic82], see also [Kar95]) noticed that the discrete quadratic variations of a semimartingale X along so called Lebesgue partitions (with vertical meshes tending sufficiently fast to 0) tend almost surely to the quadratic variation $[X]$ of X.
- In recent paper [Vov15] Vovk proved the same result for typical (in Vovk's sense) càdlàg price paths with mildly restricted jumps.

Brief history of pathwise stochastic calculus cont.

- Würmli ([Wür80]) in her Master thesis written under supervision of Hans Föllmer, defined L^2 local time of a deterministic, continuous path along some sequence of partitions and an proved analog of the **Itô-Tanaka formula**.
- In recent paper [PP15] David Prömel and Nicolas Perkowski defined for a deterministic, continuous path its local times with uniformly finite p-variation, proved analog of the Itô-Tanaka formula for these times, and finally proved the existence of local times with finite p-variation for any $p > 2$ of typical continuous price paths.
- In another recent paper [DOS18] the authors gave example of a continuous deterministic path possessing quadratic variation but no local time (along some sequence of partitions).

Brief history of pathwise stochastic calculus cont.

- In the same paper [DOS18] the authors also proved important result that for almost any Brownian path **any non-decreasing function starting from** 0 **may be its quadratic variation** (along some sequence of refining partitions).
- In [AC17] Anna Ananova and Rama Cont proved **Itô isometry** for continuous paths with continuous quadratic variation along some sequence of partitions.
- In [CP18] Rama Cont and Nicolas Perkowski introduced the notion of variation and local times of higher order and established pathwise integration and change of variable formulas for continuous paths with arbitrary regularity.
- I do not mention at all awesome developments of the rough paths theory which has much in common with the stochastic calculus of deterministic paths.

Föllmer's quadratic variation

Let $x : [0,+\infty) \to \mathbb{R}$ be a càdlàg path. In his seminal paper [F̈81] Hans Föllmer defined quadratic variation along sequence of partitions π^n, $\pi^n = \{0 = t_0^n < t_1^n < \ldots < t_{k_n}^n < \infty\}$, in the following way.
Assume that

1. $t_{k_n}^n \to +\infty$ and $\operatorname{mesh}(\pi^n) := \max_{i=1,2,\ldots,k_n}\left(t_i^n - t_{i-1}^n\right) \to 0$ as $n \to +\infty$,
2. the discrete measures $\xi^n = \sum_{i=1}^{k_n}\left(x\left(t_i^n\right) - x\left(t_{i-1}^n\right)\right)^2 \delta_{t_{i-1}}$ tend vaguely to some Radon measure ξ,
3. for any $t > 0$, $\xi(\{t\}) = (\Delta x(t))^2 = (x(t) - x(t-))^2$,

then the quadratic variation $[x]_t^\pi$ and its continuous part $[x]_t^{\pi,c}$ along sequence of partitions $\pi = (\pi^n)$ of x on the interval $[0,t]$ are defined as

$$[x]_t^\pi := \xi([0,t]), \quad [x]_t^{\pi,c} := \xi([0,t]) - \sum_{0<s\le t} (\Delta x(s))^2.$$

Föllmer's quadratic variation and Itô's formula

Since then some modifications of Föllmer's definition were introduced (see for example [Vov15], [DOS18]), since it is easy to see that no horizontal (on time axis) meshes are important but rather vertical meshes/oscillations (on value axis). All definitions lead to Itô's formula:

Theorem

Let $x : [0, +\infty) \to \mathbb{R}$ be a càdlàg path and $[x]^{\pi,c}$ be the continuous part of its quadratic variation along some sequence of partitions π. Let $F : \mathbb{R} \to \mathbb{R}$ be a function of class C^2 then for any $t > 0$

$$F(x(t)) - F(x(0)) = \int_{(0,t]} F'(x(s-))\,\mathrm{d}x(s) + \frac{1}{2}\int_{(0,t]} F''(x(s-))\,\mathrm{d}[x]_s^{\pi,c} + \sum_{0<s\le t}\left\{F(x(s)) - F(x(s-)) - F'(x(s-))\,\Delta x(s)\right\}.$$

Föllmer's quadratic variation and Itô's formula, cont.

The integral $\int_{(0,t]} F'(x(s-))\,\mathrm{d}x(s)$ is defined as

$$\int_{(0,t]} F'(x(s-))\,\mathrm{d}x(s) := \lim_{n\to+\infty}\sum_{i=1}^{k_n} F'\left(x\left(t_{i-1}^n\wedge t\right)\right)\left(x\left(t_i^n\wedge t\right)-x\left(t_{i-1}^n\wedge t\right)\right) \tag{1}$$

while the integral

$$\int_{(0,t]} F''(x(s-))\,\mathrm{d}[x]_s^{\pi,c}$$

is the usual Lebesgue-Stieltjes integral.

Föllmer's quadratic variation and Itô's formula, remarks

Having defined the quadratic variation and pathwise stochastic integral

$$\int_{(0,t]} F'(x(s-))\,\mathrm{d}x(s)$$

a fundamental question arises if it is unique. Due to Itô's formula it is equivalent with the following question.

Let $x : [0,+\infty) \to \mathbb{R}$ be a càdlàg path and assume that π and ρ are two sequences of partitions such that the quadratic variations

$$[x]^{\pi} \textbf{ and } [x]^{\rho}$$

are well defined. Are they necessary equal?

Föllmer's quadratic variation and Itô's formula, remarks

Bad news:

- It was already Paul Lévy [Lév65], who noticed that for any continuous function $x : [0,1] \to \mathbb{R}$ there exist a sequence π of nested partitions (that is $\pi^n \subset \pi^{n+1}$) of $[0,1]$, with meshes tending to 0 and such that
$$[x]_1^\pi \equiv 0.$$
- In [DOS18] Mark Davis, Jan Obłój and Pietro Siorpaes prove that for any continuous $x : [0,1] \to \mathbb{R}$, such that for any $0 \le a < b \le 1$, $V^2(x, [a,b]) = +\infty$, and **any non-decreasing** $\alpha : [0,1] \to \mathbb{R} \cup +\infty$ **starting from** 0 there exist a sequence π of nested partitions of $[0,1]$ such that for $t \in [0,1]$
$$[x]_t^\pi \equiv \alpha_t.$$

Föllmer's quadratic variation and Itô's formula, remarks

Recall that for real-valued $x : D \to \mathbb{R}$ such that $[a, b] \subset D$ we define the 2-variation (which is not the same what quadratic variation) of x on $[a, b]$ as

$$V^2\left(x, [a, b]\right) = \sup_{a \le t_0 < t_1 < \ldots < t_n \le b} \sum_{i=1}^{n} \left(x\left(t_i\right) - x\left(t_{i-1}\right)\right)^2 . \quad (2)$$

More generally, for $p > 0$ we define the p-variation of x on $[a, b]$ as

$$V^p\left(x, [a, b]\right) = \sup_{a \le t_0 < t_1 < \ldots < t_n \le b} \sum_{i=1}^{n} \left|x\left(t_i\right) - x\left(t_{i-1}\right)\right|^p .$$

If $x : [0, +\infty) \to \mathbb{R}$ is a path of a Brownian motion then (2) holds with probability 1, see [Tay72].

Föllmer's quadratic variation and Itô's formula, remarks

Good news:
Similarly as in [DOS18] one may prove that if X_t, $t \geq 0$, is càdlàg semimartingale on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$, $x = X(\omega) : [0, +\infty) \to \mathbb{R}$ is a path (for $\omega \in \Omega$) and $\pi = (\pi^n)_n$ is a sequence of (sequences) of stopping times relative to $\mathbb{F}$ such that $\pi^n = \{0 = t_0^n < t_1^n \leq \ldots\}$ and for any $T > 0$

$$O_T(x, \pi^n) := \sup\left\{|x(t) - x(s)| : s, t \in \left[t_{i-1}^n, t_i^n\right) \cap [0, T], i \in \{1, 2, \ldots\}\right\} \to 0 \text{ a.s.} \quad (3)$$

as $n \to +\infty$ then there exists some subsequence $\pi' = (\pi^{n_k})_k$ such that

$$[x]^{\pi'} = [X](\omega) \text{ a.s.}$$

Föllmer's quadratic variation and Itô's formula, remarks

Good news:
When we assume some stronger mode of convergence of $O_t(x, \pi^n)$, for example that for any $T > 0$

$$\sum_{n=1}^{+\infty} O_T(x, \pi^n) < +\infty \text{ a.s.} \tag{4}$$

then

$$[x]^\pi = [X](\omega) \text{ a.s.}$$

Similar results were proven in [GŁM18] for typical càdlàg price paths with mildly restricted downward jumps. Typical price paths are (roughly speaking) those trajectories representing possible evolution of prices of some asset which do not allow to obtain infinite wealth by risking small amount and trading this asset.

Lebesgue partitions

An example of sequence of partitions satisfying (3) are so called **Lebesgue partitions** $\pi^n = (t_k^n)_k$, which (given some sequences $c_n \to 0+$ and $r_n \in [0, c_n)$) are defined for continuous $x : [0, +\infty) \to \mathbb{R}$ in the following way: $t_0^n = 0$ and for $k = 1, 2, \ldots$

$$t_k^n = \begin{cases} \inf\left\{t \in [t_{k-1}^n, +\infty) : x(t) \in (r_n + c_n\mathbb{Z}) \setminus \left\{x\left(t_{k-1}^n\right)\right\}\right\} & \text{if } t_{k-1}^n < +\infty; \\ +\infty \text{ otherwise.} \end{cases}$$

If the sequence $(c_n)_n$ satisfies $\sum_{n=1}^{+\infty} c_n < +\infty$ then $\pi^n = (t_k^n)_k$ also satisfies (4).

In the case of càdlàg x the times t_k^n are defined in slightly more complicated way.

Another a.s. convergence to quadratic variation

In a recent paper [Łoc19] there was shown a sequence of another quantities such that if X_t, $t \geq 0$, is càdlàg semimartingale on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and $x = X(\omega) : [0, +\infty) \to \mathbb{R}$ is a path (for $\omega \in \Omega$) then they a.s. tend to the continuous part of the quadratic variation $[X]^{cont}(\omega)$. For any $c \geq 0$ and $T > 0$ we define

$$\mathrm{TV}^c(x, [0, T]) := \sup_{a \leq t_0 < t_1 < \ldots < t_n \leq b} \sum_{i=1}^{n} \max\left(\left|x(t_i) - x(t_{i-1})\right| - c, 0\right).$$

Theorem

For any sequence $c_n \to 0+$

$$c_n \cdot TV^{c_n}(x, [0, T]) \to [X]_T^{cont}(\omega) \text{ a.s.}$$

Pathwise calculus - truncated variation approach

In the same paper [Łoc19] there was also proven a version of the Itô formula for deterministic càdlàg path $x : [0, +\infty) \to \mathbb{R}$ possessing the continuous part of the quadratic variation defined as the limit

$$[x]_t^{cont} := \lim_{c \to 0+} c \cdot \mathrm{TV}^c(x, [0,t]) . \tag{5}$$

Theorem

Let $x : [0, +\infty) \to \mathbb{R}$ be a deterministic càdlàg path such that for any $t > 0$ the limit (5) exists and $\sum_{0<s\le t} \Delta(x(s))^2$ is finite. If $F : \mathbb{R} \to \mathbb{R}$ is a function of class C^2 then for any $t > 0$

$$F(x(t)) - F(x(0)) = \int_{0+}^{t} F'(x(s-))\,\mathrm{d}x(s) + \frac{1}{2}\int_{0+}^{t} F''(x(s-))\,\mathrm{d}[x]_s^{cont} + \sum_{0<s\le t} \left\{ F(x(s)) - F(x(s-)) - F'(x(s-))\,\Delta x(s) \right\} .$$

Pathwise calculus - truncated variation approach - remarks

Similarly as in Föllmer's approach, we need to define the integral $\int_{0+}^{t} F'(x(s-))\mathrm{d}x(s)$. It is defined as

$$\int_{0+}^{t} F'(x(s-))\mathrm{d}x(s) \tag{6}$$

$$:= F'(x(t))\,x(t) - F'(x(0))\,x(0) - \lim_{c\to 0+}\int_{0+}^{t} x(s-)\mathrm{d}F'(x^c(s))$$

$$- \sum_{0<s\le t} \Delta x(s)\Delta F'(x(s)),$$

where x^c is the regularization of x obtained via so called **Skorohod map on** $[-c/2, c/2]$ and the integrals $\int_{0+}^{t} x(s-)\mathrm{d}F'(x^c(s))$, $\int_{0+}^{t} F''(x(s-))\mathrm{d}[x]_s^{cont}$ are the usual Lebesgue-Stieltjes integrals.

Pathwise calculus - truncated variation approach - remarks

- Similarly as in Föllmer's approach, it is possible to define more general quadratic variations of x using more general regularizations of x, not only the Skorohod map (see [Łoc19]).
- However, the regularization of x via the Skorohod map seems to be the simplest one, similarly as the sequence of Lebesgue partitions seems to be the simplest one in Föllmer's approach.
- David Prömel and Pietro Siorpaes asked a very **natural question: is there any relationship/link between these two approaches?**

Pathwise calculus - truncated variation approach vs. Föllmer's approach

If $x : [0,+\infty) \to \mathbb{R}$ is a continuous path then the relationship between the two approaches appears to be very straightforward.
Let $\pi^{c,r}$, $c > 0$, $r \in [0,c)$ denotes the Lebesgue partition $\pi^{c,r} = \left\{0 = t_0^{c,r} < t_1^{c,r} \le \ldots\right\}$ defined as $t_0^{c,r} = 0$ and for $k = 1,2,\ldots$

$$t_k^{c,r} = \begin{cases} \inf\left\{t \in [t_{k-1}^{c,r},+\infty) : x(t) \in (r + c\mathbb{Z}) \setminus \left\{x\left(t_{k-1}^{c,r}\right)\right\}\right\} & \text{if } t_{k-1}^{c,r} < +\infty; \\ +\infty \text{ otherwise.} \end{cases}$$

We have very tight (for small cs) estimate: for any $t > 0$

$$\frac{1}{c}\int_0^c [x]_t^{\pi^{c,r}} \mathrm{d}r - 2c^2 \le c \cdot \mathrm{TV}^c(x,[0,t]) \le \frac{1}{c}\int_0^c [x]_t^{\pi^{c,r}} \mathrm{d}r. \tag{7}$$

Pathwise calculus - truncated variation approach vs. Föllmer's approach, cont.

Esimates (7) follow easily from **interval crossings representation of the truncated variation** (see [Łoc17]):

$$\mathrm{TV}^{c}(x,[0,t]) = \int_{-\infty}^{+\infty} n_t^{q,q+c}(x)\mathrm{d}q,$$

where

$$n_t^{q,q+c}(x) = \text{ number of the crossings of the (value) interval } (q, q+c) \text{ by } x \text{ during the time interval } [0,t].$$

This generalizes the *Banach Indicatrix Theorem*.

Pathwise calculus - truncated variation approach vs. Föllmer's approach, cont.

Esimates (7) imply that whenever discrete quadratic variations along all shifted Lebesgue partitions of continuous x tend to the same quadratic variation (as $c \to 0+$) then the the limit

$$\lim_{c \to 0+} c \cdot \mathrm{TV}^{c}(x, [0, \cdot])$$

also exists and it coincides with this quadratic variation.

This is also true whenever discrete quadratic variations $[x]^{\pi^{c_n,r}}$ along all shifted Lebesgue partitions tend to the same quadratic variation as $n \to +\infty$ and c_n is such that $c_n \to 0+$ and $\frac{c_{n+1}}{c_n} \to 1$.

The last property is a.s. satisfied by paths of a continuous semimartingale and for typical continuous price paths.

Pathwise calculus - truncated variation approach vs. Föllmer's approach, cont.

Using Hölder continuity of local time of a continuous semimartingale (see [RY05]) or typical continuous price path (see [PP15]), Kolmogorov continuity theorem and dowcrossing representation of a local time (see [EK78]) it possible to prove more, namely that the function

$$[0, c) \ni r \mapsto [x]^{\pi^{c,r}}$$

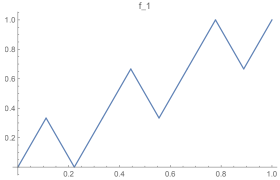
is Hölder continuous with Hölder exponent α for any $\alpha \in (0, 1/2)$ (for almost all semimartingale paths x or for typical price paths x) and with the same Hölder norm for all cs.

Pathwise calculus - truncated variation approach vs. Föllmer's approach, cont.

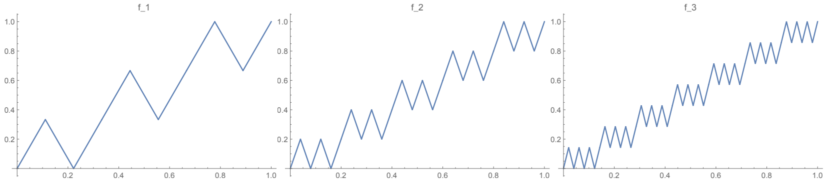
Bad news:
Unfortunately, for deterministic paths it is possible to give example of a continuous path $x : [0,1] \to \mathbb{R}$ and a sequence $c_n \to 0+$ such that

$$\lim_{n\to+\infty} [x]_1^{\pi^{c_n,0}} = 1, \quad \lim_{n\to+\infty} [x]_1^{\pi^{c_n,c_n/2}} = 0.$$

Such path may be constructed as follows. First we define sequence of functions f_n, $n = 1, 2, \dots$ which look as follows:



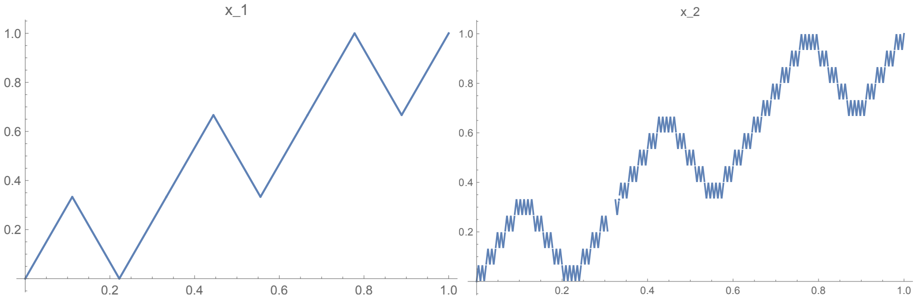
Pathwise calculus - counterexample, cont.



Next, we define

- $x_1 = f_1$,
- the graph of x_2 is obtained from the graph of x_1 by replacing each segment joining $\left(\frac{i}{9}, x_1\left(\frac{i}{9}\right)\right)$ and $\left(\frac{i+1}{9}, x_1\left(\frac{i+1}{9}\right)\right)$, $i = 0, \ldots, 8$, by (scaled, shifted and eventually reflected) graph of f_2, ...,
- the graph of x_{n+1} is obtained from the graph of x_n by replacing each segment joining $\left(\frac{i}{9 \ldots (2n+1)^2}, x_n\left(\frac{i}{9 \ldots (2n+1)^2}\right)\right)$ and $\left(\frac{i+1}{9 \ldots (2n+1)^2}, x_n\left(\frac{i+1}{9 \ldots (2n+1)^2}\right)\right)$ by the graph of f_{n+1}.

Pathwise calculus - counterexample, cont.



The sequence $(x_n)_n$ converges uniformly to a continuous x.
Now we define $c_n = \frac{1}{3 \cdot \ldots \cdot (2n+1)}$ for $n = 1, 2, \ldots$.
It is not difficult to see that (for $n = 1, 2, \ldots$) $[x]_1^{\pi^{c_n,0}} = 1$ while $[x]_1^{\pi^{c_n, c_n/2}} \le \frac{1}{2n+1}$.

Pathwise calculus - càdlàg case

With some extra effort it is possible to prove that whenever discrete quadratic variations along all shifted Lebesgue partitions of càdlàg x tend to the same quadratic variation (as $c \to 0+$) then the limit

$$\lim_{c \to 0+} c \cdot \mathrm{TV}^c(x, [0, \cdot])$$

also exists and it coincides with the continuous part of this quadratic variation.

Remark

Unfortunately, it is open question if the above assumption is satisfied a.s. by paths of a càdlàg semimartingale or by typical càdlàg price paths (with mildly restricted downward jumps).

Pathwise calculus with local times - continuous case

If $x : [0, +\infty) \to \mathbb{R}$ is a continuous path possessing quadratic variation $[x]^\pi$ along some sequence of partitions π then the local time L^π of x is (if it exists) a density of the occupation measure of x (with increments of time measured by $\mathrm{d}[x]^\pi$), more precisely, for any (Lebesgue)measurable, non-negative $m : \mathbb{R} \to [0, +\infty)$ and any $t > 0$ the following identity holds

$$\int_{\mathbb{R}} m(y) L_t^\pi(y) \mathrm{d}y = \int_0^t m(x(s)) \mathrm{d}[x]_s^\pi.$$

Following [Wür80] for $\pi^n = \{0 = t_0^n < t_1^n \le \ldots\}$ we define

$$L_t^{\pi^n}(y) := 2 \sum_{i=0}^{\infty} \mathbf{1}_{\left(x\left(t_i^n \wedge t\right) \wedge x\left(t_{i+1}^n \wedge t\right), x\left(t_i^n \wedge t\right) \vee x\left(t_{i+1}^n \wedge t\right)\right]}(y) \left| x\left(t_{i+1}^n \wedge t\right) - y \right|.$$

Pathwise calculus with local times - continuous case

Theorem (Würmli, 1980)

Let $x : [0, +\infty) \to \mathbb{R}$ be a continuous path, $\pi = (\pi^n)_n$ be a refining sequence of partitions such that for any $t > 0$, $O_t(x, \pi^n) \to 0$ and the discrete pathwise local times L^{π^n} converge weakly in $L^2(\mathrm{d}y)$ to L^π as $n \to +\infty$. Then x possess quadratic variation $[x]^\pi$ along π and for any F which is twice weakly differentiable in L^2 one has:

$$F(x(t)) - F(x(0)) = \int_{(0,t]} F'(x(s))\,\mathrm{d}x(s) + \frac{1}{2}\int_{\mathbb{R}} F''(y)\, L_t^\pi(y)\mathrm{d}y,$$

where the integral $\int_{(0,t]} F'(x(s))\,\mathrm{d}x(s)$ is defined as in (1).

Pathwise calculus with local times - càdlàg case

Definition (David Prömel, 2017)

Let $x \in D([0,T];\mathbb{R})$ and let $\pi = (\pi^n)$ be a refining sequence of partitions exhausting the jumps of x and such that for any $t > 0$, $O_t(x, \pi^n) \to 0$. A function $L^\pi \colon [0,T] \times \mathbb{R} \to \mathbb{R}$ is called L^p-local time of x along (π^n) if $L^\pi = L^{\pi,c} - 2L^{\pi,d}$ and

$$2 \sum_{t_i \in \pi^n} \mathbf{1}_{(x(t_i^n \wedge t-) \wedge x(t_{i+1}^n \wedge t), x(t_i^n \wedge t-) \vee x(t_{i+1}^n \wedge t)]}(u) |x(t_{i+1} \wedge t) - u|, \quad u \in \mathbb{R},$$

converge weakly in $L^p(\mathrm{d}u)$ to $L^{\pi,c}(u)$ as $n \to \infty$,

$$2 \sum_{t_i \in \pi^n} \mathbf{1}_{(x(t_i^n \wedge t-) \wedge x(t_i^n \wedge t), x(t_i^n \wedge t-) \vee x(t_i^n \wedge t)]}(u) |x(t_i \wedge t) - u|, \quad u \in \mathbb{R},$$

converge weakly in $L^p(\mathrm{d}u)$ to $L^{\pi,d}(u)$ as $n \to \infty$.

Pathwise calculus with local times - càdlàg case

Theorem (David Prömel, 2017)

Suppose that $x \in D([0,T];\mathbb{R})$ has a L^p-local time in the sense of the definition on the previous slide for $p>1$. If F is twice weakly differentiable in L^q with $1/p+1/q=1$, then for $t\in[0,T]$

$$F(x(t)) - F(x(0)) = \int_{(0,t]} F'(x(s))\,\mathrm{d}x(s) + \frac{1}{2}\int_{\mathbb{R}} F''(y)\,L_t^{\pi}(y)\mathrm{d}y + \sum_{0<s\le t}\left\{F(x(s)) - F(x(s-)) - F'(x(s-))\,\Delta x(s)\right\},$$

where the integral $\int_{(0,t]} F'(x(s))\,\mathrm{d}x(s)$ is defined as in (1).

Pathwise calculus with local times - càdlàg case - another approach

It appears that it is possible to obtain Tanaka-Meyer formula for deterministic càdlàg paths when local times are defined via normalized numbers of *interval/level crossings*.

Definition

Let $x \in D([0,T];\mathbb{R})$. A function $L^{cross} \colon [0,T] \times \mathbb{R} \to \mathbb{R}$ is called L^p-level crossing local time of x if it is the weak limit in L^p of the quantites

$$L_t^{cross,c}(u) = c \cdot n_t^{u,u+c}(x) \text{ as } c \to 0+,$$

where $n_t^{u,u+c}(x) =$ number of the crossings of the (value) interval $(u, u+c)$ by x during the time interval $[0,t]$.

Pathwise calculus with local times - càdlàg case - another approach

Theorem

Suppose that $x \in D([0,T];\mathbb{R})$ has a L^p-local time in the sense of the definition on the previous slide for $p > 1$. If F is twice weakly differentiable in L^q with $1/p + 1/q = 1$ and $\sum_{0<s\le t} \Delta\left(x(s)\right)^2$ is finite then for $t \in [0,T]$

$$F(x(t)) - F(x(0)) = \int_{0+}^{t} F'(x(s))\,\mathrm{d}x(s) + \frac{1}{2}\int_{\mathbb{R}} F''(y)\,L_t^{cross}(y)\mathrm{d}y$$
$$+ \sum_{0<s\le t}\left\{F(x(s)) - F(x(s-)) - F'(x(s-))\,\Delta x(s)\right\},$$

where the integral $\int_{0+}^{t} F'(x(s))\,\mathrm{d}x(s)$ is defined as in (6).

Some references I

A. Ananova and R. Cont.
Pathwise integration with respect to paths of finite quadratic variation.
J. Math. Pures Appl., 107(6):737–757, 2017.

K. Bichteler.
Stochastic integration and l^p theory of semimartingales.
Ann. Probab., 9(1):49–89, 1982.

R. Cont and N. Perkowski.
Pathwise integration and change of variable formulas for continuous paths with arbitrary regularity.
Preprint arXiv:1803.09269, 2018.

Some references II

M. Davis, J. Obłój, and P. Siorpaes.
Pathwise stochastic calculus with local times.
Ann. de l'IHP, 54(1):1–21, 2018.

N. El Karoui.
Sur les montées des semi-martingales.
Astérisque, 52–53 (Temps Locaux):63–88, 1978.

H. Föllmer.
Calcul d'Itô sans probabilités.
Séminaire de Probabilités XV, 80:143–150, 1981.

L. Galane, R. M. Łochowski, and F. J. Mhlanga.
On the quadratic variation of the model-free price paths with jumps.
Lith. Math. J., 58(2):141–156, 2018.

Some references III

R. L. Karandikar.
On pathwise stochastic integration.
Stoch. Process. Appl., 57(1):11–18, 1995.

Paul Lévy.
Processus Stochastiques et Mouvement Brownien.
Gauthier-Villars, Paris, 1965.

R. M. Łochowski.
On a generalisation of the Banach Indicatrix Theorem.
Colloq. Math., 148(2):301–314, 2017.

Some references IV

R. Łochowski.
Quadratic variation of cadlag semimartingales as a.s. limit of the normalized truncated variations.
Stochastics, accepted, 2019.

N. Perkowski and D. J. Prömel.
Local times for typical price paths and pathwise Tanaka formulas.
Electron. J. Probab., 20(46):1 – 15, 2015.

Daniel Revuz and Marc Yor.
Continuous martingales and Brownian motion, 3rd ed., volume 293 of *Grundlehren der Mathematischen Wissenschaften*.
Springer-Verlag, Berlin, 2005.

Some references V

S. J. Taylor.
Exact asymptotic estimates of Brownian path variation.
Duke Math. J., 39(2):219–241, 1972.

V. Vovk.
Itô calculus without probability in idealized financial markets.
Lith. Math. J., 55(2):270–290, 2015.

M. Würmli.
Lokalzeiten für martingale.
Master Thesis, Universität Bonn, supervised by Hans Föllmer, 1980.

Thank you for your attention!