Model-free version of the BDG inequality and its applications

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Classical and pathwise BDG inequalities

The famous Burkholder-Davis-Gundy inequalities for càdlàg martingales indexed by a continuous time parameter $t \geq 0$ read as follows

Theorem (BDG inequalities)

Let M_t, $t \geq 0$, be a (local) martingale with càdlàg paths such that $M_0 = 0$ and $p \geq 1$. Let $M_t^* := \sup_{s \leq t} |M_s|$ and $[M]$ be the quadratic variation of M. Then there exist universal (independent from M) positive constants c_p, C_p such that for any $t \geq 0$

$$c_p \mathbb{E} [M]_t^{p/2} \leq \mathbb{E} (M_t^*)^p \leq C_p \mathbb{E} [M]_t^{p/2}.$$

The classical BDG inequalities play fundamental role in the investigation of properties of stochastic integrals driven by càdlàg semimartingales.

Classical and pathwise BDG inequalities, cont.

An (almost) elemenatry proof of BDG inequalities for local martingales may be found for example in Kallenberg's book [6].

Another proof (for local martingales with continuous paths) may be based on Itô's lemma, see Revuz and Yor's monograph [9].

A very direct proof (in discrete time) follows however from the original Burkholder's method [3].

Burkholder's method

Burkholder's method (for $p = 1$) is based on a construction of some real function $W(x,t,z)$ ($x \in \mathbb{R}$, $t \geq 0$, $z \geq |x|$) satisfying the following conditions

(i) $W\left(x, x^2, |x|\right) \leq 0$,

(ii) $W(x,q,z) \geq z - C_1 q^{1/2}$,

(iii) $W\left(x+d, q+d^2, |x+d| \vee z\right) \leq W(x,q,z) + W_x(x,q,z)\,d$, $d \in \mathbb{R}$.

Let now M_t, $t = 0, 1, 2, \ldots$ be a real martingale. Using condition (iii) we get that $W\left(M_t, [M]_t, M_t^*\right)$ is a supermartingale.

Using this observation, then (ii) and (i) we immediately obtain

$$\mathbb{E}\left(M_t^* - C_1 [M]_t^{1/2}\right) \leq \mathbb{E} W\left(M_t, [M]_t, M_t^*\right) \leq \mathbb{E} W\left(M_0, [M]_0, M_0^*\right) \leq 0.$$

Classical and pathwise BDG inequalities, cont.

Naturally, it is not easy to construct W. Adam Osękowski proposes the following W:

$$W(x,q,z) := U(x,q) + U\left(z - |x|, q\right), \tag{1}$$

where

$$U(x,q) := \begin{cases} -\left(2q - x^2\right)^{1/2} \text{ if } q \geq x^2, \\ |x| - 2q^{1/2} \text{ if } q \leq x^2. \end{cases}$$

Function W satisfies condition (ii) with $C_1 = 4$ which yields the estimate

$$\mathbb{E}M_t^* \leq 4\mathbb{E}[M]_t^{1/2}.$$

Burkholder's method - more general setting

Theorem (Burkholder'02, [3])

Let B be a Banach space and suppose U and V are mappings from $B \times [0, +\infty)^2$ to $\mathbb{R}$ satisfying for $z \geq |x|$:

(a) $U(x, q, z) \geq V(x, q, z)$,

(b) $U(x, q, z) = U(x, q, |x| \vee z)$,

(c) $U\left(x + d, \sqrt{q^2 + |d|^2}, |x + d| \vee z\right) \leq U(x, q, z) + U_x(x, q, z)d$ *for any* $d \in B$

then for any B-valued martingale M_t, $t = 0, 1, 2, \ldots$,

$$\mathbb{E}V\left(M_t, [M]_t, M_t^*\right) \leq \mathbb{E}U\left(M_0, [M]_0, M_0^*\right).$$

Burkholder's method - reverse result

Theorem (Burkholder '02, [3])

Let B be a Banach space V be a mapping from $B \times [0, +\infty)^2$ to $\mathbb{R} \cup \{+\infty\}$ satisfying

$$V(x, q, z) = V(x, q, |x| \vee z).$$

Define

$$U(x, q, z) := \sup \left\{ \mathbb{E} V \left(M_t, \sqrt{q^2 - |x|^2 + [M]_t}, M_t^* \vee z \right) \right\}$$

where the supremum is taken over the set of all martingales M_t, $t = 0, 1, 2, \ldots$ starting from x and over all $t = 0, 1, 2, \ldots$. then the pair (U, V) satisfies conditions (a), (b) and (c) of the previous theorem.

Burkholder's method and beyond

U_x denotes partial derivative with respect to x or (when it is not well defined) some vector (or linear functional) from the supporting hyperplane.

In fact, Burkholder proved slightly weaker result (see [3]), where in condition (c) instead of deterministic d he has some mean-zero random variable, on the left side of condition (c) he has expectation and there is no term $U_x(x, q, z)d$ on the right side of condition (c).

The form of condition (c) presented here allows to follow a different stream of literature about martingale inequalities, which has emerged in mathematical finance, starting with Hobson's paper [5].

Hobson's approach

Let X_t, $t = 0, 1, 2, \ldots$ be a n-dimensional process of prices of n financial assets, $T \in \{1, 2, \ldots\}$ be the terminal time and let V be some option with the pay-off depending of X_T, $[X]_T$ and X_T^*. Assume that U and V satisfy conditions (a), (b) and (c).

Let us consider the trading strategy such that immediately after moment $t-1$ till the moment $t = 1, 2, \ldots, T$ an investor keeps the portfolio where the number of units of ith asset ($i = 1, 2, \ldots, n$) is equal

$$H_t^i = U_{x^i}\left(X_{t-1}, [X]_{t-1}, X_{t-1}^*\right).$$

Using (a), (b) and (c) we get that

$$U\left(X_0, [X]_0, X_0^*\right) + \sum_{i=1}^{T} H_t \cdot \left(X_t - X_{t-1}\right) \geq V\left(X_T, [X]_T, X_T^*\right).$$

Hobson's approach and Vovk's superhedging

This means that we have a trading strategy which allows to hedge the payoff of the option with the initial capital $\lambda := U(X_0, [X]_0, X_0^*)$.

Similar approach was used by Vovk in **continuous-time** setting. One more restriction imposed by Vovk [10] was that the capital process

$$\sum_{i=1}^{\infty} H_{\tau_i \wedge t} \cdot \left(X_{\tau_i \wedge t} - X_{\tau_{i-1} \wedge t}\right)$$

never drops below some threshold $-\lambda \leq 0$, thus, the investor never looses more than λ when she/he decides to stop the trading.

Vovk's approach - formal definition for continuous processes

- $d = 1, 2, \ldots$, $T > 0$ and Ω is the space of *all* continuous functions $\omega : [0, T] \to \mathbb{R}^d$.
- $S_t(\omega) = \left(S_t^1(\omega), S_t^2(\omega), \ldots, S_t^d(\omega)\right) := \omega(t) = \left(\omega^1(t), \omega^2(t), \ldots, \omega^d(t)\right)$ $t \in [0, T]$, denotes the *coordinate process*.
- $\mathcal{F} = (\mathcal{F}_t)_{t \in [0,T]}$ is the natural filtration of S.
- Stopping times $\tau : \Omega \to [0, T] \cup \{+\infty\}$ with respect to $\mathcal{F}$ and the corresponding σ-algebras $\mathcal{F}_\tau$ are defined as usual.

Vovk's approach - formal definition for continuous processes, cont.

A process $G : [0, T] \times \Omega \to \mathbb{R}^d$ is a *simple process* (*simple strategy*) if

- there exist stopping times $0 = \tau_0 \le \tau_1 \le \ldots$ and
- $\mathcal{F}_{\tau_l}$-measurable, bounded functions $g_l : \Omega \to \mathbb{R}^d$, such that for every $\omega \in \Omega$, $\tau_l(\omega) = \tau_{l+1}(\omega) = \ldots \in [0, T] \cup \{+\infty\}$ from some $l \in \{1, 2, \ldots\}$ on, and such that

$$G_t(\omega) = g_0(\omega) \mathbf{1}_{\{0\}}(t) + \sum_{l=0}^{+\infty} g_l(\omega) \mathbf{1}_{(\tau_l(\omega), \tau_{l+1}(\omega)]}(t). \qquad (2)$$

Vovk's approach - formal definition for continuous processes, cont.

The corresponding *integral process* $G \cdot S : [0, T] \times \Omega \to \mathbb{R}$,

$$(G \cdot S)_t(\omega) := \sum_{l=0}^{\infty} g_l(\omega) \cdot (S_{\tau_{l+1}(\omega) \wedge t}(\omega) - S_{\tau_l(\omega) \wedge t}(\omega))$$
$$= \sum_{l=0}^{\infty} g_l(\omega) \cdot S_{\tau_l(\omega) \wedge t, \tau_{l+1}(\omega) \wedge t}(\omega).$$

For $\lambda > 0$ a simple strategy G is called *(strongly)* λ*-admissible* if

$$(G \cdot S)_t(\omega) \geq -\lambda \text{ for all } \omega \in \Omega \text{ and all } t \in [0, T].$$

The set of strongly λ-admissible simple strategies will be denoted by $\mathcal{G}_\lambda$.

Vovk's approach - formal definition for continuous processes, cont.

Vovk's outer measure $\bar{\mathbb{P}}$ of a set $A \subseteq \Omega$ is defined as the minimal superhedging price for $\mathbf{1}_A$, that is

$$\bar{\mathbb{P}}(A) :=$$
$$\inf\left\{\lambda : \exists (G^n) \subseteq \mathcal{G}_\lambda \text{ s.t. } \forall \omega \in \Omega,\ \lambda + \liminf_{n\to\infty} (G^n \cdot S)_T(\omega) \geq \mathbf{1}_A(\omega)\right\}.$$

A property (P) holds for *typical price paths* if the set A where (P) is violated is *null set*, that is $\bar{\mathbb{P}}(A) = 0$.

Similarly, *Vovk's outer expectation* $\bar{\mathbb{E}}$ of $Z : \Omega \to \mathbb{R}$ is defined as

$$\bar{\mathbb{E}}Z :=$$
$$\inf\left\{\lambda : \exists (G^n) \subseteq \mathcal{G}_\lambda \text{ s.t. } \forall \omega \in \Omega,\ \lambda + \liminf_{n\to\infty} (G^n \cdot S)_T(\omega) \geq Z(\omega)\right\}.$$

Vovk's approach - quadratic variation and the Itô integral

- It is possible to generalize Vovk's outer measure to more general paths - namely to càdlàg paths with mildly restricted jumps (we do not precise this since we will not use this).
- Using arguments dating back at least to Kolmogorov, Vovk proved [11] that typical càdlàg price paths possess quadratic variation along some special sequence of partitions (π^n) (so called *Lebesgue partitions*), that is, that for $t \in [0, T]$ and a typical price path $\omega \in \Omega$ there exists the limit

$$[S^i, S^j]_t(\omega) := \lim_{n \to +\infty} \sum_{k=0}^{\infty} S^i_{\pi^n_k \wedge t, \pi^n_{k+1} \wedge t}(\omega) S^j_{\pi^n_k \wedge t, \pi^n_{k+1} \wedge t}(\omega).$$

Vovk's approach - quadratic variation and the Itô integral

Using the existence of the quadratic variation and the model-free Itô isometry for the integral processes driven by **continuous** price paths (F - a simple process):

$$\bar{\mathbb{P}}\left(\left\{\|(F\cdot S)\|_\infty \geq a\sqrt{b}\right\}\cap\left\{\int_0^T F_s^{\otimes 2}\mathrm{d}[S]_s \leq b\right\}\right)$$
$$\leq 2\exp(-a^2/2),$$

David Prömel and Nicolas Perkowski [8] gave a meaningful interpretation of the integral $(H \cdot S)$ for a left-continuous version H of any adapted, càdlàg process $\hat{H}$.

Vovk's approach - quadratic variation and the Itô integral

Using the existence of the quadratic variation and the pathwise BDG inequality of Mathias Bieglboeck and Pietro Siorpaes, R. Ł, David Prömel and Nicolas Perkowski [7] proved the following estimate for the integral processes driven by càdlàg price paths with **mildly restricted downward jumps** (F - a simple process):

$$\bar{\mathbb{P}}\left(\{\|(F\cdot S)\|_\infty \geq a\} \cap \{\|F\|_\infty \leq c\} \cap \{|[S]_T|_{HS} \leq b\} \cap \{\|S\|_\infty \leq M\} \right)$$
$$\leq (1+3dM+2d\psi(M))\frac{6\sqrt{b}+2+2M}{a}c,$$

which leads to a meaningful interpretation of the integral $(H\cdot S)$ for a left-continuous version H of any adapted, càdlàg process $\hat{H}$.

Vovk's approach - quadratic variation and the Itô integral

Here

$$|[S]_T|_{HS} := \left(\sum_{i,j=1}^{d} [S^i, S^j]_T^2 \right)^{1/2}$$

and $\psi : [0, +\infty) \to [0, +\infty)$ is a function which *mildly restricts* the downward jumps of ω, that is $\omega \in \Omega_\psi$ if

$$\omega^i(t-) - \omega^i(t) \leq \psi\left(\sup_{s \in [0,t)} |\omega(s)| \right) \text{ for } t \in (0, T], i = 1, \ldots, d. \quad (3)$$

Pathwise BDG inequality of Beiglböck and Siorpaes [2]

If for real $x_0, x_1, \ldots, e_0, e_1, \ldots$ and $m = 0, 1, \ldots$ we define

$$x_m^* := \max_{0 \le k \le m} |x_k|, [x]_m := x_0^2 + \sum_{k=0}^{m-1} (x_{k+1} - x_k)^2,$$

$$(e \cdot x)_m := \sum_{k=0}^{m-1} e_k (x_{k+1} - x_k)$$

then for any $p \ge 1$ there exist positive constant $C_p < +\infty$ and numbers $f_0^p, f_1^p, \ldots$ such that f_k^p, $k = 0, 1, \ldots$ depends only on $x_0, x_1, \ldots, x_k$, and such that for $N = 0, 1, \ldots$

$$(x_N^*)^p \le C_p \sqrt{[x]_N^p} + (f^p \cdot x)_N . \tag{4}$$

Pathwise BDG inequality of Beiglböck and Siorpaes, cont.

Similarly, for any $p \geq 1$ there exist positive constant $c_p < +\infty$ and numbers $g_0^p, g_1^p, \ldots$ such that g_k^p, $k = 0, 1, \ldots$ depends only on $x_0, x_1, \ldots, x_k$, and such that for $N = 0, 1, \ldots$

$$(x_N^*)^p \geq c_p \sqrt{[x]_N^p} + (g^p \cdot x)_N. \tag{5}$$

Inequalities (4) and (5) have very plausible form for $p = 1$:

$$x_N^* \leq 6\sqrt{[x]_N} + 2(h \cdot x)_N, \quad \sqrt{[x]_N} \leq 3x_N^* - (h \cdot x)_N,$$

where

$$h_n := \frac{x_n}{\sqrt{[x]_n + (x_n^*)^2}}, \text{ with the convention that } 0/0 = 0.$$

Pathwise BDG inequality of Beiglböck and Siorpaes, remarks

Remark

It is important that

$$|h_n| \leq 1,$$

since it allows to control the changes of the value of capital process which (super)hedges the pay-off x_N^*.

Remark

Osękowski's function (1) gives $h_n := W_x(x_n, [x]_n, x_n^*)$ *and for such* h_n *we have* $C_1 = 4 < 6$ *and we probably have* $|h_n| \leq 2$ *but this is to be verified.*

Model-free version of BDG inequality, some proposal

A model-free version of the BDG inequality for paths with mildly restricted downwards jumps ($\in \Omega_\psi$, i.e. satisfying (3)) would be the following:
Let H be the left continuous version of adapted càdlàg process and $p \geq 1$ then there exists $c_p, C_p \in (0, +\infty)$ such that

$$c_{p,\psi}\bar{\mathbb{E}}\left[H \cdot S\right]_t^{p/2} \leq \bar{\mathbb{E}}\left((H \cdot S)_t^*\right)^p \leq C_{p,\psi}\bar{\mathbb{E}}\left[H \cdot S\right]_t^{p/2}.$$

Let us recall that for $Z : \Omega_\psi \to \mathbb{R}$,

$$\bar{\mathbb{E}}Z :=$$
$$\inf\left\{\lambda : \exists (G^n) \subseteq \mathcal{G}_\lambda \text{ s.t. } \forall \omega \in \Omega_\psi,\ \lambda + \liminf_{n\to\infty}(G^n \cdot S)_T(\omega) \geq Z(\omega)\right\}.$$

Model-free version of BDG inequality, some proposal

Unfortunately, even for $p = 1$ and $\omega \in \Omega$ (i.e. continuous ω) it is not obvious if λ_1 and λ_2 such that for given $t \in [0, T]$

$$\exists (F^n) \subseteq \mathcal{G}_\lambda \text{ s.t. } \forall \omega \in \Omega,\ \lambda_1 + \liminf_{n\to\infty} (F^n \cdot S)_T(\omega) \geq (H \cdot S)_t^*(\omega)$$

and

$$\exists (G^n) \subseteq \mathcal{G}_\lambda \text{ s.t. } \forall \omega \in \Omega,\ \lambda_2 + \liminf_{n\to\infty} (G^n \cdot S)_T(\omega) \geq \sqrt{[H \cdot S]_t}(\omega)$$

are comparable.

If we wanted to apply the pathwise BDG inequality we would need to add a strategy which may be far below 0 at some point u between 0 and T and we do not control if the strategy G^n would compensate this.

Model-free version of BDG inequality, another proposal (for continuous price paths)

However, the BDG inequality of Beiglböck and Siorpaes may be easily applied to the following modification of Vovk's outer expectation.

Let $\mathcal{T}[0,T]$ be the family of stopping times τ such that $0 \le \tau \le T$. For any **process** $Z : [0,T] \times \Omega \to [0,+\infty]$ we define the **minimal cost of superhedging the whole process** Z as

$$\overline{\mathbb{E}} Z = \\ = \inf \left\{ \lambda : \exists \tilde{\Omega} \subset \Omega, G^n \in \mathcal{G}_\lambda \text{ s.t. } \bar{\mathbb{P}}\left(\Omega \setminus \tilde{\Omega}\right) = 0 \right. \\ \left. \text{and } \forall \omega \in \tilde{\Omega} \quad \forall \tau \in \mathcal{T}[0,T] \quad \liminf_{n \to +\infty} \left(\lambda + (G^n \cdot S)_\tau\right) \ge Z_\tau \right\}. \tag{6}$$

Model-free version of BDG inequality, another proposal - remarks

Remark

It is easy to verify that the outer expectation $\overline{\mathbb{E}}$ is countably subadditive, monotone and positively homogeneous.

Using the pathwise BDG inequality it is also not difficult to see that for any $\lambda \in \mathbb{R}$ and a simple strategy G which, starting with the initial capital λ, superhedges the process $\sqrt{[H \cdot S]_t}$, $t \in [0, T]$, the strategy

$$C_1 G + \Phi^n$$

(where $C_1 = 6$) starting with the initial capital $C_1 \lambda$ almost superhedges the process $(H \cdot S)_t^*$, $t \in [0, T]$.

Model-free version of BDG inequality, another proposal - a superhedging strategy for $(H \cdot S)^*$

Φ^n is the strategy constructed in the following way: let $\tilde{H}$ be a simple process approximating uniformly H, with the representation (2), that is

$$\tilde{H}_t(\omega) = h_0(\omega)\mathbf{1}_{\{0\}}(t) + \sum_{l=0}^{+\infty} h_l(\omega)\mathbf{1}_{(\tau_l(\omega),\tau_{l+1}(\omega)]}(t)$$

and $(\sigma_m^n)_{m\geq 0}$ - a non-decreasing rearrangement of $(\pi_k^n)_{k\geq 0} \cup (\tau_l)_{l\geq 0}$, where $(\pi_k^n)_{k\geq 0}$ is the nth Lebesgue partition ($n = 1, 2, \ldots$). We set

$$x_0^n = 0, \quad x_{m+1}^n = x_m^n + \tilde{H}_{\sigma_m^n \wedge T}(\omega) \cdot (S_{\sigma_{m+1}^n \wedge T}(\omega) - S_{\sigma_m^n \wedge T}(\omega))$$

for $m = 0, 1, 2, \ldots$. Next, for $m = 0, 1, 2, \ldots$ such that $\sigma_m^n \leq T$ we put

$$\Phi_{\sigma_m^n}^n(\omega) := 2\frac{x_m^n}{\sqrt{[x^n]_m + ((x^n)_m^*)^2}}\tilde{H}_{\sigma_m^n \wedge T}(\omega).$$

Why the strategy $C_1 G + \Phi^n$ works?

We set Φ^n to be constant and equal $\Phi^n_{\sigma^n_m}$ on the interval $\left(\sigma^n_m, \sigma^n_{m+1}\right]$.

From the pathwise BDG inequality (and the construction of Φ^n) we have that $C_1\lambda + (C_1 G + \Phi^n)\cdot S$ is at times σ^n_m (for $m = 0, 1, 2, \ldots$ such that $\sigma^n_m \le T$) is no smaller than $(x^n)^*_m - C_1\varepsilon$ as long as $\sqrt{[x^n]_m}$ does not differ from $\sqrt{\left[\tilde{H}\cdot S\right]_{\sigma^n_m}}$ by more than $\varepsilon > 0$ and $\lambda > \overline{\mathbb{E}}\sqrt{\left[\tilde{H}\cdot S\right]}$.

Next, from the fact that $\left|\Phi^n_{\sigma^n_m}\right| \le \left|\tilde{H}_{\sigma^n_m \wedge T}\right|$ and the construction of the Lebesgue partitions ($\left|S_{\pi^n_{k+1}} - S_{\pi^n_k}\right| \le 2^{-n}$ for continuous ω) it follows that the capital process $(C_1 G + \Phi^n)\cdot S$ of the strategy $C_1 G + \Phi^n$ does not diverge on the intervals $\left[\sigma^n_m \wedge T, \sigma^n_{m+1} \wedge T\right]$ too far from its values at the endpoints of these intervals.

Why the strategy $C_1 G + \Phi^n$ works?

Putting together these observations we easily obtain a sequence of strategies which, starting with the initial capital $C_1\lambda$, superhedge the process $(H\cdot S)^*_t$, $t\in[0,T]$, as long as $\lambda > \sqrt{\overline{\mathbb{E}}[H\cdot S]}$. Thus we have

Theorem

The following model-free BGD inequality for continuous model-free paths (and $p=1$) holds:

$$\overline{\mathbb{E}}(H\cdot S)^* \le C_1\overline{\mathbb{E}}\sqrt{[H\cdot S]}, \tag{7}$$

where $C_1 \le 6$. Similarly, using the second pathwise BDG inequality of [2] we obtain

$$\overline{\mathbb{E}}\sqrt{[H\cdot S]} \le C_2\overline{\mathbb{E}}(H\cdot S)^*,$$

where $C_2 \le 3$.

Multidimensional version of model-free BDG inequality

Let $\mathcal{H}$ denote the left-continuous version of an adapted processes $\hat{H} : [0, T] \times \Omega \to \mathbb{R}$.
Let H be a matrix-valued left-continuous version of an adapted process, $H : [0, T] \times \Omega \to \mathbb{R}^d \times \mathbb{R}^d = \mathbb{R}^{2d}$ and $H = \left(H^1, H^2, \ldots, H^d\right)$ where $H^i \in \mathcal{H}$, $i = 1, 2, \ldots, d$. The family of such processes will be denoted by $\mathcal{H}^d$.
For $H \in \mathcal{H}^d$ we define the integral $(H \cdot S)$ as the vector of integrals

$$(H \cdot S) = \left(\left(H^1 \cdot S\right), \left(H^2 \cdot S\right), \ldots, \left(H^d \cdot S\right)\right).$$

Also we define

$$\left|\left[(H \cdot S)\right]\right|_t := \sum_{i=1}^{d} \left[\left(H^i \cdot S\right)\right]_t.$$

Multidimensional version of model-free BDG inequality, cont.

We have the following multidimensional form of (7):

Proposition

For any $H \in \mathcal{H}^d$

$$\overline{\mathbb{E}}\,(H \cdot S)^* \leq C_1 d \overline{\mathbb{E}} \sqrt{\left|[(G \cdot S)]\right|}, \tag{8}$$

where $C_1 \leq 6$.

Spaces $\mathcal{M}$, $\mathcal{M}^d$, $loc\mathcal{M}$ and $loc\mathcal{M}^d$

Now, we introduce the space $\mathcal{M}$ (resp. $\mathcal{M}^d$) of (equivalence classes of) adapted processes $G \in \mathcal{H}$ (resp. $G \in \mathcal{H}^d$) (G is equivalent with H if $\overline{\mathbb{E}}(G-H)^* = 0$) such that $\overline{\mathbb{E}}G^* < +\infty$. Using standard arguments (see for example [8, proof of Lemma 2.11]) we see that $\mathcal{M}$ (resp. $\mathcal{M}^d$) equipped with the metric

$$d(G,H) := \overline{\mathbb{E}}(G-H)^*$$

is a complete metric space and the family of simple processes from $\mathcal{M}$ (resp. $\mathcal{M}^d$) is dense in $\mathcal{M}$ (resp. $\mathcal{M}^d$).

We also introduce the space $loc\mathcal{M}$ (resp. $loc\mathcal{M}^d$) of processes $X : [0,T] \times \Omega :\rightarrow \mathbb{R}^d$ such that for any $Q > 0$, $X^Q \in \mathcal{M}$ (resp. $X \in \mathcal{M}^d$), where

$$X_t^Q = X_t 1_{[0,Q]}(\|[S]\|_t), \quad t \in [0,T].$$

Theorem on existence and uniqueness of the solutions of SDEs with Lipschitz coefficients, driven by continuous, model-free price paths

We will assume the following:

- X_0 is such that the process $X = (X_t)_{t\in[0,T]}$ defined by $X_t = X_0$, $t \in [0, T]$, satisfies $X \in \mathcal{M}^d$;
- $K : [0, T] \times (\mathbb{R}^d)^{[0,T]} \times \Omega \to \mathbb{R}^d$ and $F : [0, T] \times (\mathbb{R}^d)^{[0,T]} \times \Omega \to \mathbb{R}^d \times \mathbb{R}^d$ are non-anticipating, by which we mean that for any adapted processes $X, Y : [0, T] \times \Omega \to \mathbb{R}^d$, $K(t, X(\omega), \omega) = K(t, Y(\omega), \omega)$ and $F(t, X(\omega), \omega) = F(t, Y(\omega), \omega)$ whenever $X_s(\omega) = Y_s(\omega)$ for all $s \in [0, t]$, and the processes $K_t(\omega) = K(t, X(\omega), \omega)$, $F_t(\omega) = F(t, X(\omega), \omega)$ are adapted (see also [4, Sect. 1]);

Theorem on existence and uniqueness of the solutions of SDEs

- $A : [0, T] \times \Omega \to \mathbb{R}$ is a stochastic process starting from 0 (i.e. $A_0 = 0$), $A = A^u - A^v$ and A^u, A^v are continuous, non-decreasing processes, starting from 0 and such for all $\omega \in \Omega$, $A^u_T(\omega) + A^v_T(\omega) \le M$, where M is a deterministic constant;
- $$\int_0^{\cdot} K(s,0,\omega)\,\mathrm{d}A^u_s, \int_0^{\cdot} K(s,0,\omega)\,\mathrm{d}A^v_s, \int_0^{\cdot} F(s,0,\omega)\,\mathrm{d}S_s(\omega) \in \mathcal{M}^d;$$
- K, F are Lipschitz in the sense that there exists $L \ge 0$ such that for all $t \in [0, T]$, $x, y : [0, T] \to \mathbb{R}^d$ and $\omega \in \Omega$

$$|K(t,x,\omega) - K(t,y,\omega)| + |F(t,x,\omega) - F(t,y,\omega)| \le L \sup_{s\in[0,t]} |x(s) - y(s)|. \tag{9}$$

Theorem on existence and uniqueness of the solutions of SDEs

Theorem

Under the assumptions stated above, the integral equation

$$X_t(\omega) = X_0(\omega) + \int_0^t K(s, X(\omega), \omega)\, \mathrm{d}A_s + \int_0^t F(s, X(\omega), \omega)\, \mathrm{d}S_s(\omega),$$

has unique solution in the space $\mathrm{loc}\mathcal{M}^d$.

The proof is standard, in the sense that it uses Picard's iterations and contraction property of appropriately defined sequence of maps $T^n : \mathcal{M}^d \to \mathcal{M}^d$. The contractivity of these maps is proved with the use of the model-free BDG inequality and the Lipschitz condition (9).

Similar results exisitng in the literature

Similar results to ours was proven by Daniel Bartl, Michael Kupper and Ariel Neufeld in [1] even for Hilbert space-valued processes, but under the assumptions that

- one can also trade the difference $\|S\|^2 - < S >$, where $\|\cdot\|$ denotes the norm in the Hilbert space and $< S >$ denotes the quadratic variation process of the coordinate process S (but defined in a different way than the usual tensor quadratic variation of a Hilbert space-valued semimartingale, see [1, Remark 2.4]),
- the measure d$< S >$ is majorized by the Lebesgue measure dt multiplied by some constant.

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Thank you!