On the Laplace transform of some functionals related to the variation of Brownian motion with drift

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Truncated variation of Brownian motion with drift

Let B_t be a standard Brownian motion and let $W_t = B_t + \mu t, t \geq 0$, be a Brownian motion with drift μ. It is well known fact that variations of B_t and W_t are infinite.

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For any $c > 0$ we define variation on interval $[a, b]$, truncated on the level c, by the following formula

$$TV_{\mu}^{c}[a; b] = \sup_{n} \sup_{a \leq t_1 < t_2 < \ldots < t_n \leq b} \sum_{i=1}^{n-1} \max\left\{ \left| W_{t_{i+1}} - W_{t_i} \right| - c, 0 \right\}.$$

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The "nature" of the defined **truncated variation** is quite different than the regular one. From uniform continuity of W_t on interval $[a, b]$ it follows that truncated variation attains its value for partition $a \leq t_1 < t_2 < \ldots < t_n \leq b$, diameter of which $\max_{1 \leq i \leq n-1}(t_{i+1} - t_i)$ is not too small and it is a.s. finite random variable.

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For $a > b$ we set $TV_\mu^c\,[a; b] = 0$.

Agenda

- Existence of exponential moments $\mathbb{E}\exp\left(\alpha TV_{\mu}^{c}[0;T]\right)$
- Universal (up to universal constants) estimates of $\mathbb{E}TV_{\mu}^{c}[0;T]$
- Two related functionals - upward truncated variation $UTV_{\mu}^{c}[0;T]$ and downward truncated variation $DTV_{\mu}^{c}[0;T]$
- Laplace transforms of UTV and DTV and their applications
- Limit distributions of UTV and DTV
- Some applications of UTV and DTV in financial mathematics.
- Open problems

Existence of exponential moments of truncated variation

For any $c > 0$ truncated variation at the level c has finite exponential moments of any order.

Let T_c be the moment of the first decrease of process W_t from its maximum to date by c:

$$T_c = \inf\{t \geq 0 : W_t \leq \sup_{0 \leq s \leq t} W_s - c\}.$$

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Further, let $T_c^{\sup}$ be the last moment, at which the maximum of W_t on interval $[0, T_c]$ is attained and let $T_c^{\inf} \leq T_c^{\sup}$ be such that $W_{T_c^{\inf}} = \inf_{0 \leq s \leq T_c^{\sup}} W_s$.

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$$TV_\mu^c [0, T] \leq W_{T_c^{\sup}} + c + TV_\mu^c [T_c, T]. \quad (1)$$

Exponential moments of truncated variation, cont.

Fix $\alpha > 0$ and let $\delta > 0$ be sufficiently small number for the following inequality to hold

$$1 - \mathbf{E} \exp \left(\alpha \sup_{0 \leq t \leq T} W_t + \alpha c \right) P \left(T_c < \delta \right) > 0.$$

Exponential moments of truncated variation, cont.

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$$1 - \mathbf{E} \exp\left(\alpha \sup_{0 \le t \le T} W_t + \alpha c\right) P\left(T_c < \delta\right) > 0.$$

Fix $M > 0$. From inequality (1) we have

$$\begin{aligned}
\mathbf{E} \exp\left(\alpha TV_\mu^c [0, T] \wedge M\right) &\le \mathbf{E} \exp\left(\alpha W_{T_c^{\sup}} + \alpha c + \alpha TV_\mu^c [T_c, T] \wedge M\right) \\
&\le \mathbf{E} \exp\left(\alpha W_{T_c^{\sup}} + \alpha c\right) \mathbf{E} \exp\left[\alpha TV_\mu^c [T_c, T] \wedge M; T_c < \delta\right] \\
&+ \mathbf{E} \exp\left(\alpha W_{T_c^{\sup}} + \alpha c\right) \mathbf{E} \exp\left[\alpha TV_\mu^c [T_c, T] \wedge M; T_c \ge \delta\right] \\
&\le \mathbf{E} \exp\left(\alpha \sup_{0 \le t \le T} W_t + \alpha c\right) \mathbf{E} \exp\left(\alpha TV_\mu^c [0, T] \wedge M\right) P\left(T_c < \delta\right) \\
&+ \mathbf{E} \exp\left(\alpha \sup_{0 \le t \le T} W_t + \alpha c\right) \mathbf{E} \exp\left(\alpha TV_\mu^c [0, T - \delta] \wedge M\right) P\left(T_c \ge \delta\right).
\end{aligned}$$

Exponential moments of truncated variation, cont.

From this we get

$$\begin{aligned}&\mathbf{E}\exp\left(\alpha TV_{\mu}^{c}[0,T]\wedge M\right)\\&\le\frac{\mathbf{E}\exp\left(\alpha\sup_{0\le t\le T}W_t+\alpha c\right)P\left(T_c\ge\delta\right)}{1-\mathbf{E}\exp\left(\alpha\sup_{0\le t\le T}W_t+\alpha c\right)P\left(T_c<\delta\right)}\mathbf{E}\exp\left(\alpha TV_{\mu}^{c}[0,T-\delta]\wedge M\right)\end{aligned}$$

Iterating the above inequality we get

$$\begin{aligned}&\mathbf{E}\exp\left(\alpha TV_{\mu}^{c}[0,T]\wedge M\right)\\&\qquad\le\left(\frac{\mathbf{E}\exp\left(\alpha\sup_{0\le t\le T}W_t+\alpha c\right)P\left(T_c\ge\delta\right)}{1-\mathbf{E}\exp\left(\alpha\sup_{0\le t\le T}W_t+\alpha c\right)P\left(T_c<\delta\right)}\right)^{\lceil T/\delta\rceil}.\end{aligned}$$

Exponential moments of truncated variation, cont.

From this we get

$$\mathbf{E}\exp\left(\alpha TV_{\mu}^{c}[0,T]\wedge M\right) \le \frac{\mathbf{E}\exp\left(\alpha\sup_{0\le t\le T}W_t+\alpha c\right)P\left(T_c\ge\delta\right)}{1-\mathbf{E}\exp\left(\alpha\sup_{0\le t\le T}W_t+\alpha c\right)P\left(T_c<\delta\right)}\mathbf{E}\exp\left(\alpha TV_{\mu}^{c}[0,T-\delta]\wedge M\right)$$

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Letting $M\to\infty$, we get $\mathbf{E}\exp\left(\alpha TV_{\mu}^{c}[0,T]\right)<+\infty$.

Universal estimates for $\mathbb{E}TV^c_\mu[0;T]$

We have

Theorem

For $c, T > 0$ and function F which will be defined later, the ratio $\frac{\mathbb{E}TV^c_\mu[0;T]}{F(\mu,c,T)}$ is separated from 0 and from ∞. In particular,one may prove

$$\frac{1}{747} \leq \frac{\mathbb{E}TV^c_\mu[0;T]}{F(\mu,c,T)} \leq 493.$$

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For $\mu = 0$ function F has the following form

$$F(0,c,T) = \begin{cases} \frac{T}{c} & \text{for } T \geq c^2, \\ (T)^{3/2} \frac{e^{-c^2/(2T)}}{c^2} & \text{for } T < c^2. \end{cases}$$

Definition of F for $\mu \neq 0$

For $|\mu|, c, T > 0$ let us define

$$\chi(c,\mu) = \sqrt{\frac{e^{2|\mu|c} - 1 - 2|\mu|c}{2\mu^2}} = c\sqrt{1 + \frac{2}{3}c|\mu| + \frac{1}{3}c^2\mu^2 + \ldots},$$

then for $\mu \neq 0$ function F has the following form

$$F(\mu, c, T) = \begin{cases} \frac{T}{c} + |\mu| T & \text{for } \sqrt{T} \geq \chi(c,\mu), \\ 2\sqrt{T} + |\mu| T - c & \text{for } c - |\mu| T \leq \sqrt{T} < \chi(c,\mu), \\ (T)^{3/2} \frac{e^{-(c-|\mu| T)^2/(2T)}}{(c-|\mu| T)^2} & \text{for } \sqrt{T} < c - |\mu| T. \end{cases}$$

Two related quantities

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- **upward truncated variation**

$$UTV_{\mu}^{c}[a,b] := \sup_{n} \sup_{a \le t_1 < s_1 < \cdots < t_n < s_n \le b} \sum_{i=1}^{n} \max\{W_{s_i} - W_{t_i} - c, 0\}$$

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- and **downward truncated variation**

$$DTV_{\mu}^{c}[a,b] := \sup_{n} \sup_{a \le t_1 < s_1 < \cdots < t_n < s_n \le b} \sum_{i=1}^{n} \max\left\{W_{t_i} - W_{s_i} - c, 0\right\}.$$

Relations between TV, UTV and DTV

We have (almost obvious) relations

$$TV_{\mu}^{c}[0,T] \geq UTV_{\mu}^{c}[0,T],$$

$$TV_{\mu}^{c}[0,T] \geq DTV_{\mu}^{c}[0,T],$$

$$TV_{\mu}^{c}[0,T] \leq UTV_{\mu}^{c}[0,T] + DTV_{\mu}^{c}[0,T],$$

$$\mathcal{L}\left(UTV_{\mu}^{c}[0,T]\right) = \mathcal{L}\left(DTV_{-\mu}^{c}[0,T]\right).$$

From these it follows e.g. that UTV and DTV have finite exponential moments.

Why one may be interested in *UTV* and *DTV*?

- We have the relations stated on the previous slide
- For *UTV* we have analog of the inequality (1), which is simply an **equality**

$$UTV_{\mu}^{c}[0,T] = \sup_{0\le t<s\le T_c \wedge T} (W_s - W_t - c)_+ + UTV_{\mu}^{c}[T_c, T], \quad (2)$$

  and due to this is possible to calculate Laplace transform of *UTV*!
- With the use of Laplace transform it is possible to estimate higher moments of *TV*
- Both quantities have some interpretation in financial mathematics

Laplace transform of UTV

From equality (2) and results of Hadjiliadis and Zhang [Hadjiliadis, Zhang 2009] it is possible to calculate Laplace transform of UTV.
Let us define

$$\begin{aligned} DD_t &= \sup_{0 \leq s \leq t} W_s - W_t, \\ DU_t &= W_t - \inf_{0 \leq s \leq t} W_t \end{aligned}$$

and

$$\begin{aligned} T_D(c) &= \inf \{ t \geq 0 | DD_t = c \}, \\ T_U(c) &= \inf \{ t \geq 0 | DU_t = c \}. \end{aligned}$$

(Note that in the just introduced notation $T_D(c)$ corresponds to T_c.)

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(Note that in the just introduced notation $T_D(c)$ corresponds to T_c.) Let us next define $p(t; a, b)\,dt := \mathbf{P}(T_D(a) \in dt, T_U(b) > t)$,
$q(t; a, b)\,dt := \mathbf{P}(T_U(a) \in dt, T_D(b) > t)$.

Laplace transform of UTV cont.

From the equation (2) it follows, that Laplace transform $L(\lambda, T) := \mathbf{E}\left(\lambda \exp UTV_{\mu}^{c}[0, T]\right)$ of UTV satisfies the following integral equation

$$L(\lambda, T) = \int_0^T \int_c^\infty e^{\lambda(y-c)} L(\lambda, T-t) \mathbf{P}\left(T_D(c) \in dt, \sup_{0 \le s \le T_D(c)} DU_s \in dy\right)$$

$$+ \int_0^T L(\lambda, T-t) \mathbf{P}\left(T_D(c) \in dt, \sup_{0 \le s \le T_D(c)} DU_s < c\right)$$

$$+ \int_c^\infty e^{\lambda(y-c)} \mathbf{P}\left(T_D(c) > T, \sup_{0 \le s \le T} DU_s \in dy\right)$$

$$+ \mathbf{P}\left(T_D(c) > T, \sup_{0 \le s \le T} DU_s < c\right). \quad (3)$$

Laplace transform of UTV cont.

The equation (3) may be stated in more elegant form using densities $p(t;a,b)$ i $q(t;a,b)$.

$$L(\lambda,T)=\int_0^T L(\lambda,T-t)\left\{p(t;c,c)+\int_c^\infty e^{\lambda(y-c)}\frac{\partial p(t;c,y)}{\partial y}dy\right\}dt$$
$$-\int_0^T \mathbf{P}(T_D(c)>T-t)\left\{q(t;c,c)+\int_c^\infty e^{\lambda(y-c)}\frac{\partial q(t;y,c)}{\partial y}dy\right\}dt$$
$$+\mathbf{P}(T_D(c)>T). \quad (4)$$

In [Hadjiliadis, Zhang 2009] there were calculated the densities $p(t;a,b)$ and $q(t;a,b)$.

Laplace transform of UTV cont.

Let us now define

$$M(\lambda,\nu) \ : \ = \int_0^{\infty} \exp(-\nu t) L(\lambda, t)\, dt,$$
$$T(\nu) \ : \ = \int_0^{\infty} \exp(-\nu t) \mathbf{P}(T_D(c) > t)\, dt.$$

From (4) we get

$$M(\lambda,\nu) = T(\nu) \frac{1 - \int_0^{\infty} e^{\nu t} q(\lambda, t)\, dt}{1 - \int_0^{\infty} e^{\nu t} p(\lambda, t)\, dt}, \tag{5}$$

where

$$p(\lambda, t) := p(t; c, c) + \int_0^{\infty} e^{\lambda y} \frac{\partial p(t; c, y + c)}{\partial y} dy,$$
$$q(\lambda, t) := q(t; c, c) + \int_0^{\infty} e^{\lambda y} \frac{\partial q(t; y + c, c)}{\partial y} dy.$$

Laplace transform of UTV cont.

From (5) we may calculate $M(\lambda, \nu)$! We have

Laplace transform of *UTV* cont.

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Theorem

For λ and ν with negative real parts the following equality holds

$$M(\lambda,\nu) = -\frac{1}{\nu} - \frac{\lambda e^{\mu c}}{\nu^2} \frac{\mu \sinh(cU_\mu(\nu)) - U_\mu(\nu)\cosh(cU_\mu(\nu))}{\frac{\lambda U_\mu(\nu)}{\nu} + \sinh(2cU_\mu(\nu)) - 2\frac{\lambda+\mu}{U_\mu(\nu)}\sinh^2(cU_\mu(\nu))}, \tag{6}$$

where $U_\mu(\nu) = \sqrt{\mu^2 - 2\nu}$.

Transforms of moments of UTV

Differentiating formula (6) for $\lambda = 0$, we get Laplace transforms of moments of UTV :

Transforms of moments of *UTV*

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$$\int_0^{\infty} e^{\nu t} \mathbf{E} UTV_{\mu}^{c}[0, t]\, dt = \frac{e^{\mu c}\sqrt{\mu^2 - 2\nu}}{2\nu^2 \sinh\left(c\sqrt{\mu^2 - 2\nu}\right)}. \qquad (7)$$

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$$\int_0^\infty e^{\nu t} \mathbf{E} \left(UTV_\mu^c [0,t]\right)^2 dt$$
$$= -\frac{e^{\mu c} U_\mu(\nu) \left[U_\mu^2(\nu) + \nu\left(1 - \cosh\left(2cU_\mu(\nu)\right)\right)\right]}{2\nu^3 \left[U_\mu(\nu)\cosh\left(cU_\mu(\nu)\right) - \mu \sinh\left(cU_\mu(\nu)\right)\right] \sinh^2\left(cU_\mu(\nu)\right)}.$$

Inverting Laplace transform - exact formulas for moments and their estimates

Inverting formula (7) one gets exact formula

$$\mathbf{E}UTV_{\mu}^{c}[0,T] = \frac{e^{\mu c}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \int_0^T (T-t) \frac{(2k+1)^2 c^2 - t}{t^{5/2}} e^{-\mu^2 t - (2k+1)^2 c^2/(2t)} dt. \qquad (8)$$

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Formula (8) may be written in the form $\mathbf{E}UTV_{\mu}^{c}[0,T]=e^{\mu c}G(|\mu|,c,T)$. Using the fact that

- $\mathbf{E}DTV_{\mu}^{c}[0,T]=\mathbf{E}UTV_{-\mu}^{c}[0,T]=e^{-\mu c}G(|\mu|,c,T)$,
- $\mathbf{E}TV_{\mu}^{c}[0,T]\sim\mathbf{E}UTV_{\mu}^{c}[0,T]+\mathbf{E}DTV_{\mu}^{c}[0,T]$,

and comparing with the estimate $\mathbf{E}TV_{\mu}^{c}[0,T]$, we get

$$F(|\mu|,c,T)\sim(e^{\mu c}+e^{-\mu c})G(|\mu|,c,T)\sim e^{|\mu|c}G(|\mu|,c,T).$$

Inverting Laplace transform - exact formulas for moments and their estimates, cont.

From this we get

$$G(|\mu|, c, T) \sim e^{-|\mu|c} F(|\mu|, c, T)$$

and

$$\mathbf{E} UTV_{\mu}^{c}[0, T] = e^{\mu c} G(|\mu|, c, T) \sim e^{\mu c - |\mu| c} F(|\mu|, c, T)$$

$$= e^{\mu c - |\mu| c} \begin{cases} T/c + |\mu| T & \text{if } \sqrt{T} \geq \chi(c, \mu); \\ 2\sqrt{T} + |\mu| T - c & \text{if } \sqrt{T} \in (c - |\mu| T, \chi(c, \mu)); \\ T^{3/2} \frac{\exp(-(c - |\mu| T)^2 / (2T))}{(c - |\mu| T)^2} & \text{if } \sqrt{T} \leq c - |\mu| T, \end{cases}$$

where $\chi(c, \mu) = \sqrt{\frac{e^{2\mu|c|} - 2\mu|c| - 1}{2\mu^2}} = c\sqrt{1 + \frac{2}{3}|\mu| c + \ldots} \geq c.$

Some remark about technique

Calculation of the transform $L(\lambda, T) := \mathbf{E}\left(\lambda \exp UTV_{\mu}^{c}[0, T]\right)$ is much simpler, when time T is random.
In fact, transform $M(\lambda, \nu) = \int_0^{\infty} \exp(-\nu t) L(\lambda, t)\, dt$ is a transform of $UTV_{\mu}^{c}[0, T]/\nu$, fot T being exponentially distributed r. v. with expected value $1/\nu$.

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Using the same technique Salminen and Valois [Salminen, Vallois, 2007] calculated the joint distribution of *maximal drawup*

$$D_t = \sup_{0 \le u < v \le t} (W_v - W_u)$$

and *maximal drawdown*

$$U_t = \sup_{0 \le u < v \le t} (W_u - W_v),$$

for $t \sim \mathrm{Exp}(\nu)$.

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$$U_t = \sup_{0 \le u < v \le t} (W_u - W_v),$$

for $t \sim \text{Exp}(\nu)$. Based on this they calculated $\text{Cor}(D_1, U_1) \approx -0.47936$ (problem stated by Gabor Szekely).

Limit distributions of UTV and DTV

Using (2), results from [Hadjiliadis, Vecer, 2006] and theory of renewal processes one may derive limit distributions of UTV and DTV for $T \to +\infty$ and (which seems to be much more interesting) for $c \to 0$.

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$$\left(UTV_{\mu}^{c}[0, T] - \frac{\mu}{1 - e^{-2\mu c}} T \right) / \sqrt{T}$$

tends in law to normal distribution

$$\mathcal{N}\left(0, \frac{e^{4\mu c} - 4e^{2\mu c}\mu c - 2\mu^2 c^2 - 2e^{-2\mu c}\mu c - e^{-2\mu c}}{\left(e^{2\mu c} - 1 - 2\mu c\right)\left(1 - e^{-2\mu c}\right)^2}\right),$$

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which in case of $\mu = 0$ simplifies to $\mathcal{N}(0, \frac{11}{12})$.

Limit distributions of UTV and DTV, cont.

For $c \to 0$ we obtain that the variable

$$UTV_{\mu}^{c}[0, T] - \left(\frac{1}{3}\mu + \frac{1}{2c}\right) T$$

tends in law to normal distribution

$$\mathcal{N}\left(0, \frac{11}{12} T\right).$$

Hence we observe that variance of the truncated variation when $c \to 0$ does not change!

Limit distributions of UTV and DTV, cont.

For $c \to 0$ we obtain that the variable

$$UTV_{\mu}^{c}[0,T] - \left(\frac{1}{3}\mu + \frac{1}{2c}\right)T$$

tends in law to normal distribution

$$\mathcal{N}\left(0, \frac{11}{12}T\right).$$

Hence we observe that variance of the truncated variation when $c \to 0$ does not change!
Limit distributions of UTV may be calculated from identities
$\mathcal{L}\left(DTV_{\mu}^{c}[0,T]\right) = \mathcal{L}\left(UTV_{-\mu}^{c}[0,T]\right).$

Some application of UTV in financial mathematics

Assumptions:

- Dynamics of the prices of some financial instrument P follows geometric Brownian motion process

$$dP_t = \mu P_t dt + \sigma P_t dB_t.$$

- For every transaction one has to pay commission proportional to the value of the transaction
- $\gamma \in (0, 1)$ is the ratio of the commission to the value of the transaction

Some application of *UTV* in financial mathematics

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Corollary: Maximal return from trading P on time interval $[0, T]$ equals

$$\exp(UTV^{c/\sigma}_{\mu/\sigma-\sigma/2}[0;T]) - 1,$$

where $c = \ln \frac{1+\gamma}{1-\gamma}$.

Sketch of the proof

Let $0 \leq t_1 < s_1 < ... < t_n < s_n \leq T$,
t_i - moments of purchasing P,
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Price of P at the moment t equals $P_t = \exp(\mu t - \sigma^2 t/2 + \sigma B_t)$ and the return from trading P reads as $\prod_{i=1}^{n} \left\{ \frac{P_{s_i}}{P_{t_i}} \frac{1-\gamma}{1+\gamma} \right\} - 1$.
Denote $\tilde{\mu} = \mu - \sigma^2/2$ and let M_n be family of partitions
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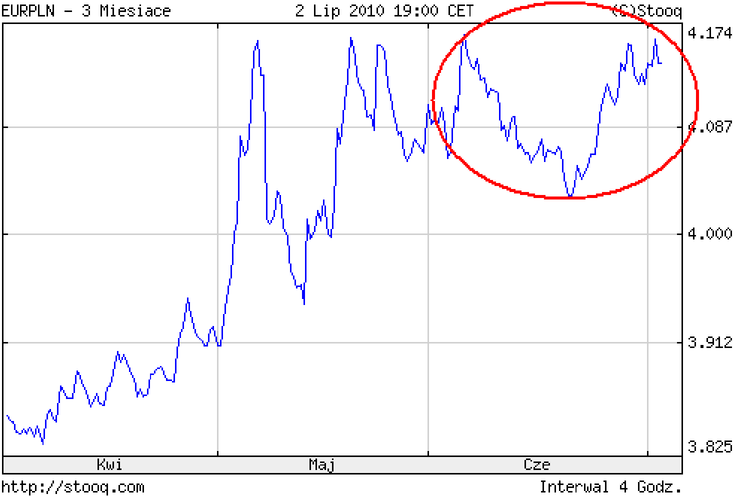
$$\sup_n \sup_{M_n} \prod_{i=1}^n \left\{ \frac{P_{s_i}}{P_{t_i}} \frac{1-\gamma}{1+\gamma} \right\} = \sup_n \sup_{M_n} \prod_{i=1}^n \left\{ \frac{\exp\left(\tilde{\mu} s_i + \sigma B_{s_i}\right)}{\exp\left(\tilde{\mu} t_i + \sigma B_{t_i}\right)} e^{-c} \right\}$$

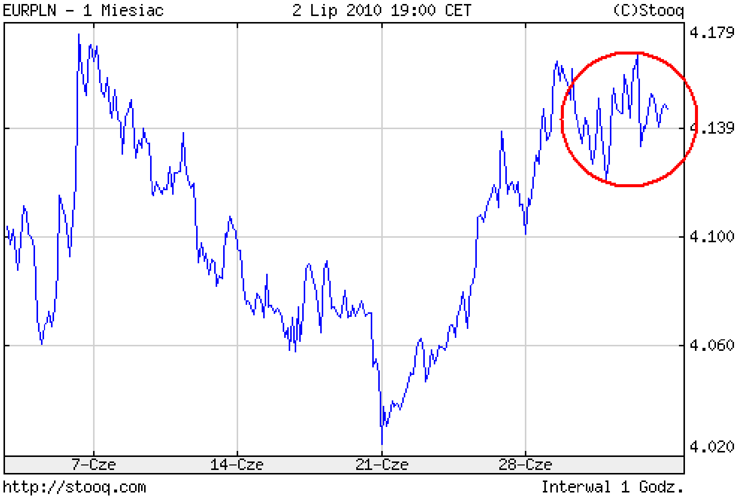
$$= \sup_n \sup_{M_n} \exp\left(\sigma \sum_{i=1}^n \left\{ \left(\frac{\tilde{\mu}}{\sigma} s_i + B_{s_i} \right) - \left(\frac{\tilde{\mu}}{\sigma} t_i + B_{t_i} \right) - \frac{c}{\sigma} \right\} \right)$$

$$= \exp\left(\sigma UTV_{\tilde{\mu}/\sigma}^{c/\sigma} [0, T] \right) = \exp\left(\sigma UTV_{\mu/\sigma - \sigma/2}^{c/\sigma} [0, T] \right).$$

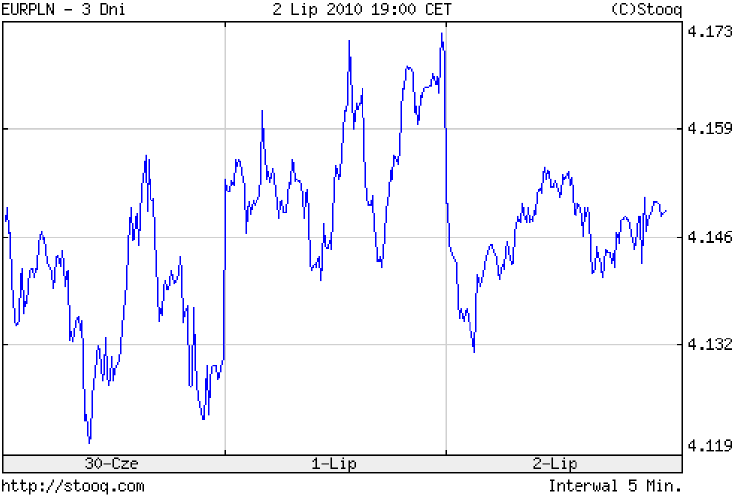
Application of DTV as a risk measure



Application of *DTV* as a risk measure, cont.



Application of *DTV* as a risk measure, cont.



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Thank you!