On pathwise stochastic integration with finite variation processes uniformly approximating càdlàg processes

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Truncated variation - how it appears

Let $f : [a; b] \to \mathbb{R}$ be a càdlàg function and let $c > 0$
Question: what is the smallest total variation possible of a (càdlàg) function from the ball $\left\{g : \|f - g\|_\infty \le \frac{1}{2}c\right\}$?

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$$\inf_{g:\|f-g\|_\infty \le \frac{1}{2}c} TV(g,[a;b]) \ge TV^c(f,[a;b]),$$

where

$$TV(g,[a;b]) := \sup_n \sup_{a\le t_0<t_1<\ldots<t_n\le b} \sum_{i=1}^n |g(t_i)-g(t_{i-1})|,$$

$$TV^c(f,[a;b]) := \sup_n \sup_{a\le t_0<t_1<\ldots<t_n\le b} \sum_{i=1}^n \max\{|f(t_i)-f(t_{i-1})|-c,0\},$$

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and follows immediately from the inequality

$$\left|g\left(t_i\right) - g\left(t_{i-1}\right)\right| \geq \max\left\{\left|f\left(t_i\right) - f\left(t_{i-1}\right)\right| - c, 0\right\}.$$

Truncated variation - definition and another interpretation

We will call the quantity

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truncated variation of the function f at the level c.

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truncated variation of the function f at the level c.
Trunated variation may be interpreted not only as the lower bound obtained on the previous slide but also as the variation taking into account only jumps greater than c.
It also possible to show that in fact we have the equality

$$\inf_{g:\|f-g\|_\infty \le \frac{1}{2}c} TV(g,[a;b]) = TV^c(f,[a;b]).$$

For $f : [0;+\infty) \to \mathbb{R}$ we will denote

$$TV^c(f,[0;t]) =: TV^c(f,t).$$

Assymptotic properties of truncated variation

It appears that the truncated variation is closely related with $p-$variation, defiend as

$$V^{p}(f,t)=\sup_{n}\sup_{0\leq t_0<t_1<\ldots<t_n\leq t}\sum_{i=1}^{n}\left|f(t_i)-f(t_{i-1})\right|^{p}.$$

Let

- $\mathcal{V}_p$ be the class of functions $f:[0;+\infty)\to\mathbb{R}$ with locally finite $p-$variation and
- $\mathcal{U}_p$ be the class of such functions f that for any $t>0$,

$$\limsup_{c\downarrow 0} c^{p-1}TV^{c}(f,t)\in(0;+\infty)$$

In [5] it is shown that for any $p\geq 1$ and $\delta>0$ we have inclusions $\mathcal{V}_p\subset\mathcal{U}_p\subset\mathcal{V}_{p+\delta}$ and for $p>1$ these inclusions are strict.

Limit distributions of truncated variation processes of Brownian motion with drift as $c \to 0$

Let $X_t = B_t$, $t \geq 0$, be a standard Brownian motion. Since it has infinite total variation on any interval $[0; t]$, $t > 0$, for any $t > 0$

$$\lim_{c \downarrow 0} TV^c(X, t) = \infty.$$

It may be of interest (due to the geometric interpretation of truncated variation and its assymptotic properties) to investigate the rate of $TV^c(X, t)$ for small cs. The answer is the following

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Theorem ([4])

For any $T > 0$ the process $c \cdot TV^c(X, t)$ converges almost surely in $(\mathcal{C}[0; T], \mathbb{R})$ topology to the deterministic function $id : [0; T] \to \mathbb{R}, id(t) = t$.

Generalisation for continuous semimartingales

Let now $X_t, t \geq 0$, be continuous a semimartingale, with the decomposition $X_t = X_0 + M_t + A_t$, with M_t being a local martingale and A_t being a continuous process with finite total variation.
Using the inequality $\max\{|x+y| - c, 0\} \leq \max\{|x| - c, 0\} + |y|$ we obtain, that

$$TV^c(X_t, t) \leq TV^c(M_t, t) + TV^0(A_t, t)$$

and

$$TV^c(M_t, t) \leq TV^c(X_t, t) + TV^0(-A_t, t).$$

Hence we conclude easily that

$$\lim_{c \downarrow 0} c \cdot TV^c(X, t) = \lim_{c \downarrow 0} c \cdot TV^c(M, t)$$

whenever any of the above limits exists.

Generalisation for continuous semimartingales, cont.

We will use the Theorem from the previous slide, Dambis and Dubins-Schwarz Theorem saying that every continuous, local martingale $M_t, t \geq 0$, with $M_0 = 0$ and infinite total variation may be represented as

$$M_t = B_{\langle M,M \rangle_t},$$

where B_t is a standard Brownian motion and the fact that the truncated variation does not depend on continuous and strictly increasing change of time variable. Utilizing above facts, we obtain that

$$\lim_{c \downarrow 0} c \cdot TV^c(X, t) = \lim_{c \downarrow 0} c \cdot TV^c(M, t) = \lim_{c \downarrow 0} c \cdot TV^c(B, \langle M, M \rangle_t) = \langle M, M \rangle_t .$$

Noticing that $\langle X, X \rangle_t = \langle M, M \rangle_t$, we finally obtain

$$\lim_{c \downarrow 0} c \cdot TV^c(X, t) = \langle X, X \rangle_t .$$

Concentration properties

Truncated variation seems to be more informative than $p-$variation:

- the finitness of $p-$ variation may be recovered form the assymptotic properties of the truncated variation;
- for fixed $c > 0$, one may look at TV^c as a random variable with the natural geometric interpretation mentioned.

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- for fixed $c > 0$, one may look at TV^c as a random variable with the natural geometric interpretation mentioned.

It appears that for fixed $c > 0$ and for X being a fBm or a diffusion with moderate growth, TV^c reveals strong concentration properties. For example, for a standard Brownian motion we have

Theorem ([1])

For the standard Brownian motion $X = B$, $t \cdot c^{-1}$ is comparable up to a universal constant with $\mathbf{E}TV^c(X,t)$ and for some universal constants A, B the Gaussian concentration holds

$$\mathbf{P}(TV^c(B,t) \geq A \cdot t \cdot c^{-1} + B\sqrt{t}u) \leq \exp(-u^2), \quad \text{for } u \geq 0.$$

The Skorohod problem

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The truncated variation also appears to be related with the Skorohod problem on $[-c/2; c/2]$. Let

- $D[0; +\infty)$ be a set of real-valued, **càdlàg functions**, defined on the interval $[0; +\infty)$,
- $BV[0; +\infty)$ denote a subset of $D[0; +\infty)$ consisting of **functions with locally finite total variation** and
- $I[0; +\infty)$ denote a subset of $D[0; +\infty)$ consisting of **non-decreasing functions**.

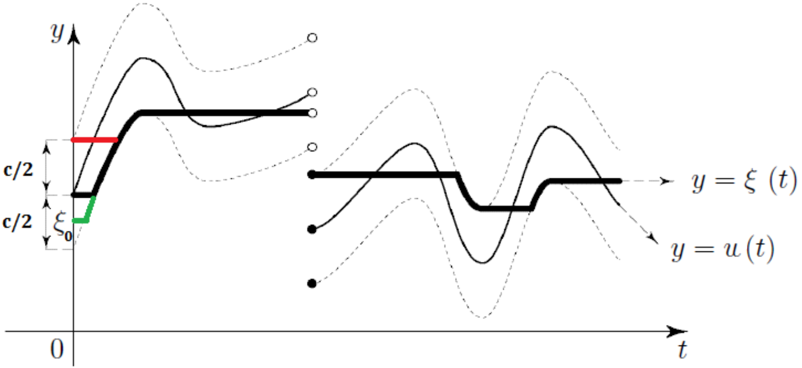
The Skorohod problem, cont.

A pair of functions $(\phi, -\xi) \in D[0;+\infty) \times BV[0;+\infty)$ is said to be a solution of **the Skorohod problem** on $[-c/2, c/2]$ with starting condition $\xi(0) = \xi^0$ for $u \in D[0;+\infty)$ if the following conditions are satisfied:

1. for every $t \geq 0$, $\phi(t) = u(t) - \xi(t) \in [-c/2, c/2]$;
2. $\xi = \xi_u - \xi_d$, where $\xi_u, \xi_d \in I[0;+\infty)$ and the corresponding measures $\mathrm{d}\xi_u$, $\mathrm{d}\xi_d$ are carried by $\{t \geq 0 : \phi(t) = c/2\}$ and $\{t \geq 0 : \phi(t) = -c/2\}$ respectively;
3. $\xi(0) = \xi^0$.

For $\xi^0 \in [u(0) - c/2; u(0) + c/2]$ the Skorohod problem has a unique solution.

Graphical interpretation of the Skorohod problem



Source: Pavel Krejˇc'ı, *Long-time behaviour of solutions to hyperbolic equations with hysteresis*, WIAS, Berlin,

The Skorohod problem and the truncated variation

Let u^{c,ξ^0} be the solution of the Skorohod problem with starting condition $\xi^0 \in [u(0) - c/2; u(0) + c/2]$.
For any such ξ^0 and $t > 0$ we have

$$TV^c(u,t) \le TV\left(u^{c,\xi^0}, t\right) \le TV^c(u,t) + c.$$

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For simplicity let us set $u^c = u^{c,u(0)}$. From the properties of the Skorohod map, we get

$$\int_0^t (u - u^c)\mathrm{d}u^c = \int_0^t \frac{c}{2}\mathrm{d}u_u^c - \int_0^t -\frac{c}{2}\mathrm{d}u_d^c = \frac{c}{2} \cdot TV\left(u^c, t\right),$$

where u_u^c and u_d^c are non-decreasing functions from the definition of the Skorohod problem, such that $u^c = u^c(0) + u_u^c - u_d^c$.

Wong-Zakai's pathwise approach to the stochastic integral

Since many years probabilists tried to define the stochastic integral in a pathwise way.

One of the earliest of such attemps is due to Wong and Zakai (1965). For $T > 0$ they considered the following approximation of Brownian paths:

(A) for all $t \in [0; T]$, $B_t^n \to B_t$ pointwise as $n \uparrow +\infty$, where B^n, $n = 1, 2, \ldots,$ are continuous and have locally bounded variation;

(B) (A) and there exists such a locally bounded process Z that for all $t \in [0; T]$, $|B_t^n| \le Z_t$;

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Theorem ([6])

Let $\psi(t, x)$ has continuous partial derivatives $\frac{\partial \psi}{\partial t}$ and $\frac{\partial \psi}{\partial x}$ and let B^n satisfy (B), then for the Lebesgue-Stieltjes integrals $\int_0^T \psi(t, B_t^n) \mathrm{d}B_t^n$, a.s.,

$$\lim_{n\to\infty} \int_0^T \psi(t, B_t^n) \mathrm{d}B_t^n = \int_0^T \psi(t, B_t) \mathrm{d}B_t + \frac{1}{2}\int_0^T \frac{\partial \psi}{\partial x}(t, B_t) \mathrm{d}t.$$

Disadvantages of the Wong-Zakai construction

- The Wong-Zakai construction can not be extended when one considers **different approximating sequences** of the underlying Brownian motion in integrator and integrand.
- Using properties of the Skorohod map and the truncated variation it is relatively easy to construct an appropriate example.

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Set: $Y^n := B^{1/n^2} + n\left(B^{1/(2n^2)} - B^{1/n^2}\right)$. We easily check that Y^n and $Z^n := B^{1/n^2}$ satisfy (A)-(B) for B. We have $B^{c/2} - B^c \geq c/4$ on the set $\mathrm{d}B^c > 0$ and $B^{c/2} - B^c \leq -c/4$ on the set $\mathrm{d}B^c < 0$. Thus

$$\int_0^1 Y^n \mathrm{d}Z^n - \int_0^1 Z^n \mathrm{d}Z^n = \int_0^1 n\left(B^{1/(2n^2)} - B^{1/n^2}\right)\mathrm{d}B^{1/n^2}$$

$$\geq n\frac{1}{4n^2}\int_0^1 \left|\mathrm{d}B^{1/n^2}\right| = \frac{n}{4}n^{-2}TV\left(B^{1/n^2},1\right) \geq \frac{n}{4}n^{-2}TV^{1/n^2}(B,1).$$

The Skorohod problem approximating sequence

Let X_t, $t \geq 0$, be a process with càdlàg paths. The process X^c obtained via the Skorohod map has the following properties:

(i) X^c has locally finite total variation;

(ii) X^c has càdlàg paths;

(iii) for every $T \geq 0$

$$|X_t - X_t^c| \leq \frac{1}{2}c;$$

(iv) for every $T \geq 0$

$$|\Delta X_t^c| \leq |\Delta X_t|,$$

where $\Delta X_t^c = X_t^c - X_{t-}^c$, $\Delta X_t = X_t - X_{t-}$;

(v) the process X^c is adapted to the natural filtration of X.

A generalisation of the Skorohod problem approximating sequence

In [3] for any càdlàg process X the small generalisation is considered. For any $c > 0$ we consider a process X^c such that

(i) X^c has locally finite total variation;

(ii) X^c has càdlàg paths;

(iii) for every $T \geq 0$ there exists such $K_T < +\infty$ that for every $t \in [0; T]$,

$$|X_t - X_t^c| \leq K_T c;$$

(iv) for every $T \geq 0$ there exists such $L_T < +\infty$ that for every $t \in [0; T]$,

$$|\Delta X_t^c| \leq L_T |\Delta X_t|,$$

where $\Delta X_t^c = X_t^c - X_{t-}^c$, $\Delta X_t = X_t - X_{t-}$;

(v) the process X^c is adapted to the natural filtration of X.

A modification of the Wong-Zakai construction, cont.

Further, in [3] the following theorems were proven.

Theorem

If processes X and Y are càdlàg semimartingales then for the sequence of the pathwise Lebesgue-Stieltjes integrals $\int_0^T Y_- \mathrm{d}X^c$ we have

$$\int_0^T Y_- \mathrm{d}X^c \to^{\mathbb{P}}_{c\downarrow 0} \int_0^T Y_- \mathrm{d}X + [X, Y]_T^{cont}.$$

$\int_0^T Y_- \mathrm{d}X$ denotes here the (semimartingale) stochastic integral and $[X, Y]^{cont}$ denotes here the continuous part of $[X, Y]$, i.e.

$$[X, Y]_T^{cont} = [X, Y]_T - \sum_{0<s\leq T} \Delta X_s \Delta Y_s.$$

A modification of the Wong-Zakai construction, cont.

Moreover

Theorem

When $c(n) > 0$ *and* $\sum_{n=1}^{+\infty} c(n)^2 < +\infty$ *then we have*

$$\int_0^T Y_- \mathrm{d}X^{c(n)} \to \int_0^T Y_- \mathrm{d}X + [X, Y]_T^{cont} \text{ a.s.}$$

Drawbacks of the construction presented

Unfortunately, the construction presented does not work for any càglàd integrand Y.

It is possible to construct a **continuous, globally bounded, adapted to the natural Brownian filtration** process Y and a sequence $B^{c(n)}$, $n = 1, 2, \ldots$, satisfying all conditions (i)-(v) for $X = B$ such that the integral

$$\int_0^1 Y \mathrm{d} B^{c(n)}$$

diverges.

First (cf. [3]) we define sequence $b(n)$, $n = 1, 2, \ldots$ in the following way $b(1) = 1$ and for $n = 2, 3, \ldots$

$$b(n) = n^2 b(n-1)^6.$$

Now we define $a(n) := b(n)^{1/2}$, $c(n) := b(n)^{-1}$ and set

$$Y := \sum_{n=2}^{\infty} a(n)\left(B - B^{c(n)}\right).$$

The proof also utilises the concentration properties of TV^c.

Bichteler's construction

The remarkable Bichteler's approach provides pathwise construction for integration of any adapted càdlàg process Y with càdlàg semimartingale integrator X and is based on the approximation

$$\lim_{n\to\infty} \sup_{0\le t\le T} \left| Y_0 X_0 + \sum_{i=1}^{\infty} Y_{\tau_{i-1}^n \wedge t} \left(X_{\tau_i^n \wedge t} - X_{\tau_{i-1}^n \wedge t} \right) - \int_0^t Y_- \mathrm{d}X \right| = 0 \text{ a.s.},$$

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where $\tau^n = (\tau_i^n)$, $i = 0, 1, 2, \ldots,$ is the following sequence of stopping times: $\tau_0^n = 0$ and for $i = 1, 2, \ldots,$

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Remark

In fact, given $c(n) > 0$, $\sum_{n=1}^{\infty} c^2(n) < +\infty$, *Bichteler's construction works for any sequence* $\tau^n = (\tau_i^n)$, $i = 0, 1, 2, \ldots$, *of stopping times, such that* $\tau_0^n = 0$ *and for* $i = 1, 2, \ldots$, $\tau_i^n = \inf \left\{ s > \tau_{i-1}^n : \left| Y_s - Y_{\tau_{i-1}^n} \right| \ge c(n) \right\}$.

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Thank you!