On pathwise stochastic integration

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The Riemann-Stieltjes integral

The **Riemann-Stieltjes integral** of a deterministic function $f : [a, b] \to \mathbb{R}$ (*integrand*) with respect to another deterministic function $g : [a, b] \to \mathbb{R}$ (*integrator*) is defined as the limit of sums

$$\sum_{i=1}^{n} f(s_i)\{g(t_i) - g(t_{i-1})\},$$

where $s_i \in [t_{i-1}; t_i]$, $\pi = \{a = t_0 < t_1 < ... < t_n = b\}$, as the mesh of the partition π, $\mathrm{mesh}(\pi) := \max_{i=1,2,...,n} |t_i - t_{i-1}|$, goes to 0.

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The Riemann-Stieltjes integral, denoted by

$$(RS)\int_{[a;b]} f \, \mathrm{d}g$$

may not exist for a given pair (f, g). The simplest situation when it happens is when the **total variation** of g is infinite.

The total variation and the existence of the R-S integral

Recall that the **total variation** of a deterministic function $g : [a, b] \to \mathbb{R}$ is defined as

$$TV\left(g,[a;b]\right) := \sup_n \sup_{a \le t_0 < t_1 < \ldots < t_n \le b} \sum_{i=1}^{n} \left| g\left(t_i\right) - g\left(t_{i-1}\right) \right|,$$

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The finiteness of $TV(g, [a; b])$ together with the continuity of f guarantee the existence of $(RS) \int_{[a;b]} f \mathrm{d}g$.
However, still, for a bounded, Borel-measurable integrand f and finite (total) variation integrator g, $(RS) \int_{[a;b]} f \mathrm{d}g$ may not exist.
This may happen e.g. when the jumps of the function f coincide with the jumps of g (the invention of an appropriate example might be a good, instructive exercise for students).

The refinement of the R-S integral - the Lebesgue-Stieltjes integral

A fine refinement of the R-S integral is the Lebesgue-Stieltjes integral. The construction is made by the introduction of the measure space $([a;b], \mathcal{B}([a;b]), \mu_g)$, where $\mathcal{B}([a;b])$ denotes the σ-field of Borel subsets of $[a;b]$ and μ_g is a *signed*, σ-finite measure on $[a;b]$.
To define μ_g we consier the càdlàg version of g (right-continuous with left limits) which we will also denote by g and for $a \le c \le d \le b$ define

$$\mu_g(c;d] := g(d) - g(c).$$

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Now we extend the measure μ_g to all Borel subsets of $[a;b]$ (Caratheodory's extension) and define the Lebesgue-Stieltjes integral as the usual Lebesgue integral of f with respect to the measure μ_g :

$$(LS)\int_{[a;b]} f\,\mathrm{d}g := \int_{[a;b]} f\,\mathrm{d}\mu_g.$$

The Lebesgue-Stieltjes integral - properties

Now, finiteness of $TV(g, [a; b])$ together with boundedness and Borel-measurability of the integrand f **guarantee** the existence of $(LS) \int_{[a;b]} f \mathrm{d}g$.

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If both, f and g are càdlàg and have finite total variation we have important *integration by parts* formula

$$
\begin{aligned}
(LS) \int_{(a;b]} f(t-) \mathrm{d}g(t) \; = & f(b)g(b) - f(a)g(a) \\
& -(LS) \int_{(a;b]} g(t-) \mathrm{d}f(t) - \sum_{a<s\leq b} \Delta f(s) \Delta g(s).
\end{aligned} \tag{1}
$$

where $\Delta f(s) = f(s) - f(s-)$, $\Delta g(s) = g(s) - g(s-)$.

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$$\begin{aligned}(LS)\int_{(a;b]} f(t-)\mathrm{d}g(t) &= f(b)g(b) - f(a)g(a) \\ &- (LS)\int_{(a;b]} g(t-)\mathrm{d}f(t) - \sum_{a<s\le b} \Delta f(s)\Delta g(s).\end{aligned} \quad (1)$$

where $\Delta f(s) = f(s) - f(s-)$, $\Delta g(s) = g(s) - g(s-)$.
Notice, that using this formula one may propose a reasonable value for $(LS)\int_{(a;b]} f(t-)\mathrm{d}g(t)$ whenever f and g are càdlàg, $TV(f, [a;b]) < +\infty$ and g is bounded! (The same observation may be used to differentiate distributions but it is another story...)

The Lebesgue-Stieltjes integral still insufficient

(A standard) Brownian motion B is one of the simplest continuous-time process with continuous trajectories, widely used in stochastic modelling and optimisation.
Unfortunately, its paths have a.s. infinite total variation. When one wants to integrate locally finite variation deterministic function or locally finite variation stochastic process with respect to a Brownian trajectory B_t, $t \in [0; T]$, one may use the relation (1) (this idea is due to Paley, Wiener and Zygmund).

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Unfortunately, this is still not sufficient to calculate (or at least to give a reasonable meaning if the calculations were too hard) e.g.

$$(LS) \int_{(a;b]} B_t \mathrm{d} B_t.$$

Functions with finite and infinite total variation - examples

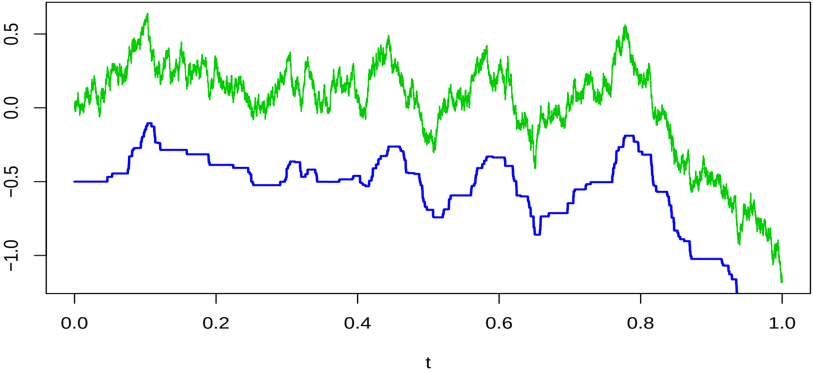

Figure : A typical path of a Brownian motion (green) and a function with finite total variation (blue)

A standard Brownian motion - an "axiomatic" definition

A **standard Brownian** motion is a stochastic process $B_t,\ t \geq 0$ defined on some (rich enough) probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the following properties

- $B_0 \equiv 0$;
- B has continuous paths, i.e. the probability $\mathbb{P}$ that for $\omega \in \Omega$, the path

  $$[0; +\infty) \ni t \mapsto B_t(\omega) \in \mathbb{R}$$

  is continuous equals 1;
- for any $0 \leq s < t < u$ increments $B_u - B_t$ and $B_t - B_s$ are independent;
- for any $0 \leq s < t$ the increment $B_t - B_s$ has normal distribution with mean 0 and variance $t - s$, $\mathcal{N}(0, t - s)$.

Finiteness of the quadratic variation of a Brownian motion

Cruicial observation which is sometimes utilised to define a stochastic integral with respect to the Brownian motion or even with respect to much more general family of processes - *semimartingales* - is the finiteness of their *quadratic variation*.

But even with the quadratic variation we must be careful. When we define

$$V_2(B,[a;b]) := \sup_n \sup_{a \le t_0 < t_1 < \ldots < t_n \le b} \sum_{i=1}^{n} \left(B_{t_i} - B_{t_{i-1}}\right)^2,$$

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then $V_2(B,[a;b])$ is a.s. infinite.

However, one may relatively easily prove that for any sequence of partitions $\pi^k = \left\{a = t_0^k < t_1^k < \ldots < t_{n(k)}^k = b\right\}$, with $\text{mesh}(\pi^k) \downarrow 0$ as $k \uparrow +\infty$ we have

$$\sum_{i=1}^{n(k)} \left(B_{t_i^k} - B_{t_{i-1}^k}\right)^2 \to^{\mathbb{P}} [B]_b - [B]_a := b - a,$$

where $\to^{\mathbb{P}}$ denotes the convergence in probability.

ψ- variation of a Brownian motion

For $x > 0$ define

$$\psi(x) := \frac{x^2}{\ln(\ln(1/x) \vee 2)}$$

and $\psi(0) := 0$ then, by the result of S. J. Taylor (*Exact asymptotic estimates of Brownian path variation*, Duke Math. J. 39), almost surely:

$$V_\psi(B, [a; b]) := \sup_n \sup_{0 \le t_0 < t_1 < \ldots < t_n \le T} \sum_{i=1}^{n} \psi\left(\left|B_{t_i} - B_{t_{i-1}}\right|\right) < +\infty, \quad (2)$$

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moreover, ψ is a function with the greatest possible order at 0, for which (2) holds.

Remark

Moreover, when mesh $\left(\pi^{(n)}\right) \to 0$, *then*

$$\lim_{n \to \infty} \sup_n \sup_{0 \le t_0 < t_1 < \ldots < t_n \le T} \sum_{i=1}^{n} \psi\left(B_{t_i^{(n)}} - B_{t_{i-1}^{(n)}}\right) = T \ a.s.$$

A small detour - why the quadratic variation $[B]$ of the Brownian motion on the interval $[a; b]$ equals $b - a$?

To give a hint why the quadratic variation of the Brownian motion on the interval $[a; b]$ equals $b - a$ let us recall the simplest construction of a standard Brownian motion (due to Donsker we know that this construction works, though the convergence is relatively slow).

- Pick (very) large integer n;
- set $t = 0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n}{n}, \frac{n+1}{n}, ...; \mathrm{d}t = \frac{1}{n}$;
- set $B_0 = 0$ and for $t = 0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n}{n}, \frac{n+1}{n}, ...$

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$$B_{t+\mathrm{d}t} = B_{t+1/n} = B_t + \begin{cases} \frac{1}{\sqrt{n}} & \text{with probability } 1/2; \\ -\frac{1}{\sqrt{n}} & \text{with probability } 1/2. \end{cases}$$

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Now notice that $(\mathrm{d}B_t)^2 = \mathrm{d}t$ (the simplest version of Itô's formula) and thus

$$\sum_{t=1/n,2/n,\ldots,T} (\mathrm{d}B_t)^2 = \sum_{t=1/n,2/n,\ldots,T} \mathrm{d}t = [B]_T = T.$$

Another small detour - Young's integral

When the integrand has finite p-variation, V_p the integrator has finite q-variation, V_q, $p > 1$, $q > 1$ and $1/p + 1/q > 1$, then one still may define

$$(RYS) \int_{[a;b]} f \mathrm{d}g.$$

where $(RYS) \int$ denotes some refinement of the Riemann-Stieltjes integral (the refinement is made to avoid problems with discontinuities) and when g is continuous, it coincides with the Riemann-Stieltjes integral.

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where $(RYS) \int$ denotes some refinement of the Riemann-Stieltjes integral (the refinement is made to avoid problems with discontinuities) and when g is continuous, it coincides with the Riemann-Stieltjes integral.
This is proved with the following Love-Young inequality:

$$\left| \sum_{i=1}^{n} f(s_i) \{g(t_i) - g(t_{i-1})\} - f(s_{i_0}) \{g(b) - g(a)\} \right| \leq \zeta \left(p^{-1} + q^{-1}\right),$$

for any partition $\pi = \{a = t_0 < t_1 < ... < t_n = b\}$, $s_i \in [t_{i-1}; t_i]$, $i, i_0 \in \{1, 2, \ldots, n\}$. Here

$$\zeta(r) = \sum_{k=1}^{\infty} k^{-r}.$$

The Young integral is still insufficient

Since the ψ- variation of a Brownian motion is finite then any p- variation with $p > 2$ of the Brownian motion is locally finite. (Just to make it precise let us write a formula for p-variation:

$$V_p\left(B,[a;b]\right) := \sup_n \sup_{0\le t_0<t_1<\ldots<t_n\le T} \sum_{i=1}^{n} \left|B_{t_i} - B_{t_{i-1}}\right|^p.$$

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) Thus, the Young integral may be used for the pathwise integration

$$(RYS) \int_{[a;b]} X_s \mathrm{d} B_s$$

whenever X is a stochastic process with locally finite q-variation, where $1 \le q < 2$.
Unfortunately, this is still unsufficient for calcuation of the integral $\int_{[a;b]} B_s \mathrm{d} B_s$.

Stochastic integral - the quadratic variation approach

For simplicity we will fix on integration with respect to the Brownian motion B.
We consider a family of stochastic processes $\mathcal{X}$ with càglàd (left continuous with right limits) paths $[0; +\infty) \ni t \mapsto X_t \in \mathbb{R}$ such that for every process $X \in \mathcal{X}$ the following hold

1. for every $u > t \geq 0$ the increment $B_u - B_t$ is independent of the values of X_s, $0 \leq s \leq t$;
2. $\|X\|_{\mathbb{H}} := \mathbb{E} \int_0^{+\infty} X_s^2 \mathrm{d}[B]_s = \mathbb{E} \int_0^{+\infty} X_s^2 \mathrm{d}s = \int_0^{+\infty} \mathbb{E} X_s^2 \mathrm{d}s < +\infty.$

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It appears that $\mathbb{H} = (\mathcal{X}, \|\cdot\|_{\mathbb{H}})$ is a Hilbert space and the family of *simple processes* of the form

$$K = K_{-1} \cdot \mathbf{1}_0 + \sum_{i=0}^{+\infty} K_i \cdot \mathbf{1}_{(t_i, t_{i+1}]},$$

where $0 = t_0 < t_1 < t_2 < \ldots$ with $t_i \uparrow +\infty$ as $i \uparrow +\infty$ and $B_u - B_{t_i}$, $u \geq t_i$ being independent from K_i is dense in $\mathbb{H}$.

Stochastic integral - the quadratic variation approach cont.

Now, for any $t > 0$ and every simple process K we define

$$\int_0^t K \mathrm{d}B := K_{-1} \cdot B_0 + \sum_{i=0}^{n-1} K_i \cdot \left(B_{t_{i+1}} - B_{t_i}\right) + K_n \cdot (B_t - B_{t_n}), \qquad (3)$$

whenever $t_n \leq t < t_{n+1}$.

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whenever $t_n \le t < t_{n+1}$.
By this definition and the independence of the increments $B_{t_{i+1}} - B_{t_i}$ from K_i we easily calculate

$$\mathbb{E}\left(\int_0^t K \mathrm{d}B\right)^2 \le \mathbb{E}\left(\int_0^{+\infty} K \mathrm{d}B\right)^2 = \|K\|_{\mathbb{H}}^2. \qquad (4)$$

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The equality (4) is called **Itô's isometry**. Now, for any $X \in \mathcal{X}$ and $\varepsilon > 0$ we find a simple process K_ε such that $\|X - K_\varepsilon\|_{\mathbb{H}} < \varepsilon$ and define

$$\int_0^t X \mathrm{d}B = \lim_{\varepsilon \downarrow 0} \int_0^t K_\varepsilon \mathrm{d}B.$$

By (4) this limit exists and is unique.

Stochastic integral - the quadratic variation approach cont.

The construction presented on two previous slides may be extended to *martingales*, *local martingales* and *semimartingales*.

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A continuous-time **martingale** M_t, $t \geq 0$, is a special (càdlàg) process, conditional increments of which, $M_t - M_s$, $0 \leq s < t$, with respect to the available information - "filtration" till moment s, $\mathcal{F}_s$, are centered

$$\mathbb{E}\left[M_t - M_s \middle| \mathcal{F}_s\right] = 0.$$

Every such a process has finite quadratic variation $[M]$. Moreover, the process $M^2 - [M]$ is also a martingale.

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A **local martingale** is a (càdlàg) process, which stopped at the appropriate *stopping times* is a *martingale*.
A **semimartingale** is a (càdlàg) process which is a sum of a local martingale and a locally finite variation process.

Semimartingales

The family of semimartingales is rich enough to encompass majority of processes used in stochastic modelling (except maybe fractional Brownian motions).
Moreover, the deep result of Bichteler and Dellacherie states that this family is in some sense the broadest possible family of good integrators.
Namely, when we define for a given integrator M and every simple process the integral $\int_0^t K \mathrm{d}M$ with the formula analogous to the formula (3), then the transformation

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To deal with continuity we need to define topologies: the space of simple processes is endowed with uniform convergence in (t, ω) topology and the space of integrals is topologized by convergence in probability.

Few properties of a stochastic integral

The stochastic integral is now defined for any semimartingale M and any (predictable) càglàd process X. For example $\int_0^t B_t \mathrm{d}B_t = \frac{1}{2}B_t^2 - \frac{1}{2}t$.
We have important

Theorem (counterpart of the Lebesgue dominated convergence)

For any càglàd (predictable) processes $X^1, X^2, \ldots,$ such that $X^i \le |X|$ for $i = 1, 2, \ldots,$ where X is some càglàd (predictable) process and $\lim_{i\uparrow+\infty} X^i \to 0$ pointwise then we have

$$\sup_{0\le s\le T} \left| \int_0^s X^i \mathrm{d}M \right| \to^{\mathbb{P}} 0.$$

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Theorem (counterpart of the Lebesgue dominated convergence)

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$$\sup_{0\le s\le T} \left| \int_0^s X^i \mathrm{d}M \right| \to^{\mathbb{P}} 0.$$

For any $p > 0$ we also have important Burkholder-Davis-Gundy's inequality

$$\mathbb{E} \sup_{0\le s\le T} \left| \int_0^s X \mathrm{d}M \right|^p \le C_p \cdot \mathbb{E} \left| \int_0^T X^2 \mathrm{d}[M] \right|^{p/2}.$$

Wong-Zakai's pathwise approach to the stochastic integral

Since many years mathematicians tried to define the stochastic integral in a pathwise way.

One of the earliest of such attemps is due to Wong and Zakai (1965). For $T > 0$ they considered the following approximation of Brownian paths:

(A) for all $s \in [0; T]$, $B_s^n \to B_s$ pointwise as $n \uparrow +\infty$, where B^n, $n = 1, 2, \ldots$, are continuous and have locally bounded variation;

(B) (A) and there exists such a bounded process Z that for all $s \in [0; T]$, $|B_s^n| \leq Z_s$;

and stated the following approximation theorem.

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Theorem (Wong-Zakai (1965))

Let $\psi(t,x)$ has continuous partial derivatives $\frac{\partial \psi}{\partial t}$ and $\frac{\partial \psi}{\partial x}$ and let B^n satisfy (B), then for the Lebesgue-Stieltjes integrals $\int_0^T \psi(t, B_t^n) \, \mathrm{d}B_t^n$, a.s.,

$$\lim_{n\to\infty} \int_0^T \psi(t, B_t^n) \, \mathrm{d}B_t^n = \int_0^T \psi(t, B_t) \, \mathrm{d}B_t + \frac{1}{2} \int_0^T \frac{\partial \psi}{\partial x}(t, B_t) \, \mathrm{d}t.$$

Wong-Zakai's pathwise approach to the stochastic integral, cont.

The correction term in Wong-Zakai's theorem, $\int_0^T \frac{\partial \psi}{\partial x}(t, B_t)\,\mathrm{d}t$, is simply the *quadratic covariation* of the processes B_t and $\psi(t, B_t)$.
For two semimartingales X and Y the **quadratic covariation**, $[X, Y]$, is the (unique) limit in the probability of the sums

$$\sum_{i=1}^{n(k)} \left(X_{t_i^k} - X_{t_{i-1}^k} \right) \cdot \left(Y_{t_i^k} - Y_{t_{i-1}^k} \right),$$

where the sequence of partitions $\pi^k = \left\{ a = t_0^k < t_1^k < \ldots < t_{n(k)}^k = b \right\}$ is such that $\text{mesh}(\pi^k) \downarrow 0$ as $k \uparrow +\infty$.

Wong-Zakai's pathwise approach to the stochastic integral, cont.

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For two càdlàg semimartingales X and Y the integral

$$(S)\int_0^T Y_- \mathrm{d}X := \int_0^T Y_- \mathrm{d}X + \frac{1}{2}[X, Y]$$

is called the *Stratonovich integral*.

Disadvantages of the Wong-Zakai construction

- The Wong-Zakai construction works only for very limited family of integrators: Brownian motions and processes of the form $X_t = \int_0^t \mu_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s$ and integrands must be functions of the integrators.
- The generalisation for any pair of semimartingale integrator and integrand is impossible.
- It is relatively easy to give an example of two sequences of continuous, with locally finite variation, bounded (and adapted to the natural Brownian filtration) processes $\tilde{B}^n$ and B^n such that $\tilde{B}^n \rightrightarrows B$, $B^n \rightrightarrows B$ uniformly but

$$\int_0^1 \tilde{B}^n \mathrm{d}B^n$$

diverges (Lochowski (2013)).

A modification of the Wong-Zakai construction

In Lochowski (2013) for any càdlàg process X the following construction is considered. For any $c > 0$ we there exists a process X^c such that

(i) X^c has locally finite total variation;

(ii) X^c has càdlàg paths;

(iii) for every $T \geq 0$ there exists such $K_T < +\infty$ that for every $t \in [0; T]$,

$$|X_t - X_t^c| \leq K_T c;$$

(iv) for every $T \geq 0$ there exists such $L_T < +\infty$ that for every $t \in [0; T]$,

$$|\Delta X_t^c| \leq L_T |\Delta X_t|,$$

where $\Delta X_t^c = X_t^c - X_{t-}^c$, $\Delta X_t = X_t - X_{t-}$;

(v) the process X^c is adapted to the natural filtration of X.

A modification of the Wong-Zakai construction, cont.

Further, in Lochowski (2013) it was shown that if processes X and Y are càdlàg semimartingales then the sequence of pathwise Lebesgue-Stieltjes integrals

$$\int_0^T Y_- \mathrm{d}X^c \to^{\mathbb{P}}_{c \downarrow 0} \int_0^T Y_- \mathrm{d}X + [X, Y]_T^{cont}.$$

$\int_0^T Y_- \mathrm{d}X$ denotes here the (semimartingale) stochastic integral and $[X, Y]^{cont}$ denotes here the continuous part of $[X, Y]$, i.e.

$$[X, Y]_T^{cont} = [X, Y]_T - \sum_{0<s\le T} \Delta X_s \Delta Y_s.$$

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$$[X,Y]_T^{cont} = [X,Y]_T - \sum_{0<s\leq T} \Delta X_s \Delta Y_s.$$

Moreover, when $c(n) > 0$ and $\sum_{n=1}^{+\infty} c(n)^2 < +\infty$ then the convergence of $\int_0^T Y_- \mathrm{d}X^{c(n)}$ to $\int_0^T Y_- \mathrm{d}X + [X,Y]_T^{cont}$ holds almost surely.

Drawbacks of the construction presented

Unfortunately, the construction presented does not work for any càglàd integrand Y.

It is possible to construct a continuous, bounded (and adapted to the natural Brownian filtration) process Y and a sequence $B^{c(n)}$, $n = 1, 2, \ldots$, satisfying all conditions (i)-(v) for $X = B$ such that the integral

$$\int_0^1 Y \mathrm{d} B^{c(n)}$$

diverges (cf. Lochowski (2013)).

Bichteler's construction

The remarkable Bichteler's approach provides pathwise construction for integration of any adapted càdlàg process Y with càdlàg semimartingale integrator X and is based on the approximation

$$\lim_{n\to\infty} \sup_{0\le s\le T} \left| Y_0 X_0 + \sum_{i=1}^{\infty} Y_{\tau_{i-1}^n \wedge s} \left(X_{\tau_i^n \wedge s} - X_{\tau_{i-1}^n \wedge s} \right) - \int_0^s Y_- \mathrm{d}X \right| = 0 \text{ a.s.},$$

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where $\tau^n = (\tau_i^n)$, $i = 0, 1, 2, \ldots$, is the following sequence of stopping times: $\tau_0^n = 0$ and for $i = 1, 2, \ldots$,

$$\tau n = \inf \left\{ t > \tau_{i-1}^n : \left| Y_t - Y_{\tau_{i-1}^n} \right| \ge 2^{-n} \right\}.$$

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Remark

In fact, given $c(n)>0$, $\sum_{n=1}^{\infty}c^2(n)<+\infty$, Bichteler's construction works for any sequence $\tau^n=(\tau_i^n)$, $i=0,1,2,\ldots,$ of stopping times, such that $\tau_0^n=0$ and for $i=1,2,\ldots,$ $\tau_i^n=\inf\left\{t>\tau_{i-1}^n:\left|Y_t-Y_{\tau_{i-1}^n}\right|\ge c(n)\right\}$.

Norvaiša's integral for functions with finite quadratic variation

In a long (171 pages!) paper, Norvaiša develops (among other results) a theory of an integral for deterministic functions with finite λ - quadratic variation.

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A function $f : [a; b] \to \mathbb{R}$ has finite λ - quadratic variation if the sums

$$\sum_{i=1}^{n(k)} \left(f(t_i^k) - f(t_{i-1}^k) \right)^2$$

converge for any nested partitions i.e. $\pi^k \subset \pi^{k+1}$ with $t_0^k = a$, $t_{n(k)}^k = b$ and mesh $\left(\pi^k\right) \to 0$ as $k \uparrow +\infty$.

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Sample paths of a Brownian motion have this property with probability $\mathbb{P}$ equal 1 - a result due to Paul Lévy.

Norvaiša's integral for functions with finite quadratic variation, cont.

Further, he defines *Left-Cauchy λ integral* of g with respect to f, $(LC)\int_a^b g\mathrm{d}f$, as a limit of the sums

$$\sum_{i=1}^{n(k)} g(t_{i-1}^k)\left(f(t_i^k) - f(t_{i-1}^k)\right)$$

and proves its existence for the integrands of the form $g(t) = \psi(f(t))$, where ψ is a C^1 function.

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He manages to prove a version of Itô's formula for this integral. But still this is far from the theory of an integral for any (locally bounded) deterministic integrands and finite quadratic variation integrators.

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The reason for this might be the fact that the analogous construction for semimartingales does not require only finiteness of the quadratic variation of the integrator but also a centering property of the increments of a local martingale part.

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Thank you!