Lévy processes - a broad class of processes used in financial modelling

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Stochastic processes - mathematical tools for modelling the evolution of phenomena with uncertain outcomes

- **Stochastic processes** are flexible mathematical tools for modelling the evolution of phenomena with uncertain outcomes
- Examples of such phenomena are: stock prices, numbers of cases of some disease in a given area, the level of Nile river etc.
- In a most general setting, stochastic process is simple a collection of **random variables**

  $$X_t : \Omega \to E, \quad t \in T,$$

  where Ω is a probability space (equipped with a **probability function** $\mathbb{P} : \Omega \to [0, 1]$), E is the space of possible values of X_t and T is some set
- For given $\omega \in \Omega$ the **random function**

  $$T \ni t \mapsto X_t(\omega) \in E$$

  is called the **trajectory** of the process X

Lévy processes - definition

A **Lévy process** on $\mathbb{R}^d$, starting from some $\mathbf{x} \in \mathbb{R}^d$ is a collection of random variables $X_t : \Omega \to \mathbb{R}^d$, $t \in [0, +\infty)$, which has the following properties

- $X_0 = \mathbf{x}$ almost surely (with probability 1);
- the **increments of** X **are independent**, this means that for any $0 \le s < t < u$ the variables $X_u - X_t$ and $X_t - X_s$ are independent;
- the **increments of** X **are stationary**, this means that for any $0 \le s < t$ and $\Delta > 0$ the variables $X_{t+\Delta} - X_t$ and $X_{s+\Delta} - X_s$ are identically distributed (have the same probability laws). In other words, for any (Borel) set $A \subset \mathbb{R}^d$,

$$\mathbb{P}\left(X_{t+\Delta} - X_t \in A\right) = \mathbb{P}\left(X_{s+\Delta} - X_s \in A\right);$$

- last but not least, the process X is **continuous in probability**, which means that for any $r > 0$, $\lim_{t \to 0+} \mathbb{P}\left(|X_t| > r\right) = 0$.

Lévy processes - another (more user-friendly ;-) definition

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$$\mathbb{P}\left(X_{t+\Delta} - X_t \in A\right) = \mathbb{P}\left(X_{s+\Delta} - X_s \in A\right);$$

- last but not least, the trajectories of the process X are almost surely **càdlàg = right-continuous with left limits** .

Financial modeling with Lévy processes - examples

One of the first models used in financial mathematics incorporating Lévy processes was Merton's jump-diffusion model (1976). He modeled the dynamics of a stock price S_t by an SDE of the type

$$\mathrm{d}S_t = S_{t-}\left(a\mathrm{d}t + \sigma \mathrm{d}B_t + \mathrm{d}L_t\right) \text{ for } t \geq 0 \text{ and } S_0 > 0. \tag{1}$$

In the above B denotes a standard Brownian motion and $L = \sum_{n=1}^{N_t} Y_n$ denotes a compound Poisson process (it will be defined later). Solution to (1) is given by

$$S_t = S_0 \exp\left(\left(a - \frac{\sigma^2}{2}\right) t + \sigma B_t\right) \prod_{n=1}^{N_t} \left(1 + Y_n\right).$$

An important generalisation of Merton's jump-diffusion model is given by an exponential of the form

$$S_t = S_0 \exp\left(X_t\right),$$

where X denotes a Lévy process.

Lévy processes - examples

The class of Lévy processes is very broad.

- However, it appears that the only **Lévy process** on $\mathbb{R}^d$ with **continuous trajectories** is a Brownian motion with drift, which means that

$$X_t = A \cdot B_t + M \cdot t, \quad t \geq 0,$$

  where A is a $d \times d$ real matrix, $M \in \mathbb{R}^d$ is a drift and B is a standard Brownian motion on $\mathbb{R}^d$.
  Recall that for a standard Brownian motion on $\mathbb{R}^d$ and $0 \leq s < t$ one has that $B_t - B_s$ has normal distribution with mean 0 and covariance matrix $(t-s)I_d$, where I_d is the d-dimesional identity matrix.
- Another fundamental example of a Lévy process is a **Poisson process**. It attains values on $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Poisson process - properties

The Poisson process N with intensity λ has the following properties

- it has independent and stationary increments (like any Lévy process);
- $N_0 = 0$ with probability 1;
- for any $0 \le s < t$ the difference $N_t - N_s$ has Poisson distribution with parameter (expectation) $\lambda(t-s)$. From this it follows that

$$\mathbb{P}\left(N_t - N_s = k\right) = e^{-\lambda(t-s)} \frac{\left(\lambda(t-s)\right)^k}{k!}, \quad k = 0, 1, 2, \ldots;$$

- in particular, from the first three properties it folows that the jumps of the process N have always size 1 and occur at the random times $\tau_1, \tau_2, \tau_3, \ldots$. Setting $\tau_0 = 0$ we have that the differences $\tau_k - \tau_{k-1}$, $k = 0, 1, 2, \ldots$, are independent and

$$\mathbb{P}\left(\tau_k - \tau_{k-1} > t\right) = e^{-\lambda t}, \quad k = 0, 1, 2, \ldots,$$

thus they have exponential distribution with parameter λ.

Poisson process - construction

It is not obvious that the Poisson process exists.

A sketch of its construction is the following. Let $T > 0$ be a (large) fixed number.

- We take $X_1, X_2, \ldots$ i.i.d. ($=$ independent and identically distributed) random variables uniformly distributed on $[0, T]$.
- Next, let $N(T)$ be a Poisson random variable with mean λT, independent from $X_1, X_2, \ldots$.
- Now, for $t \in (0, T]$ we set

$$N_t := \sum_{i=1}^{N(T)} \mathbf{1}_{\{X_i \in [0,t]\}}.$$

- To define N_t for $t \in (T, 2T]$ we take $\tilde{X}_1, \tilde{X}_2, \ldots$ i.i.d. r. vs uniformly distributed on $[T, 2T]$ and a Poisson random variable $\tilde{N}(T)$ with mean λT, independent from $X_1, X_2, \ldots, \tilde{X}_1, \tilde{X}_2, \ldots$ and $N(T)$. Next, for $t \in (T, 2T]$ we set $N_t := N(T) + \sum_{i=1}^{\tilde{N}(T)} \mathbf{1}_{\{\tilde{X}_i \in [T,t]\}}$.

Compound Poisson process

Any Poisson process has very simple structure of jumps - they are always equal 1.

What happens if we allow jumps of any size ???

It appears that if N is a Poisson process, $Y, Y_1, Y_2, \ldots$ are i.i.d. real random variables ($Y_1, Y_2, \ldots$ represent the sizes and direction of consecutive jumps), independent also from N, then setting

$$X_t = \sum_{k=1}^{N_t} Y_k$$

we obtain another Lévy process called **compound Poisson process**. From the properties of N it is easy to infer that X is also a Lévy process.

Remark

Negative Y_i corresponds to a negative jump at time τ_i.

Compound Poisson processes - building blocks of Lévy processes

Compound Lévy processes allow for much greater flexibility than Poisson processes.

It appears that they (together with the Brownian motion with drift) are building blocks of all other Lévy processes.

The big variety of compound Poisson processes makes them difficult to investigate. For example, it is quite challenging to find the distribution of X_t for fixed t.

One of the most useful tools to deal with such processes are **characteristic functions**. The characteristic function of the variable X_t is defined as

$$\varphi_t(u) = \mathbb{E}e^{iuX_t}.$$

There is a fundamental 1-1 correspondence between characteristic functions and distributions of r. vs.

Characteristic functions of Poisson and compound Poisson processes

To warm up let us first calculate the characteristic function of a Poisson process with inensity λ.

$$\mathbb{E}e^{iuN_t} = \sum_{k=0}^{\infty} e^{iuk}\mathbb{P}\left(N_t = k\right) = \sum_{k=0}^{\infty} e^{iuk}e^{-\lambda t}\frac{(\lambda t)^k}{k!}$$

$$= e^{-\lambda t}\sum_{k=0}^{\infty}\frac{\left(\lambda t e^{iu}\right)^k}{k!} = e^{-\lambda t}e^{\lambda t e^{iu}} = e^{\lambda t\left(e^{iu\cdot 1}-1\right)}.$$

Next, for the compound Poisson process $X_t = \sum_{k=1}^{N_t} Y_k$ we calculate

$$\mathbb{E}e^{iuX_t} = \sum_{k=0}^{\infty}\mathbb{E}e^{iu\sum_{n=1}^{k}Y_n}\mathbb{P}\left(N_t = k\right) = \sum_{k=0}^{\infty}\left(\mathbb{E}e^{iuY}\right)^k e^{-\lambda t}\frac{(\lambda t)^k}{k!}$$

$$= e^{-\lambda t}\sum_{k=0}^{\infty}\frac{\left(\lambda t\mathbb{E}e^{iuY}\right)^k}{k!} = e^{-\lambda t}e^{\lambda t\mathbb{E}e^{iuY}} = e^{\lambda t\mathbb{E}\left(e^{iuY}-1\right)}.$$

Characteristic functions of compound Poisson processes - cont.

In the calculation of the characteristic function of the compound Poisson process we used fundamental property of characteristic functions: if Y and Z are independent real random variables then

$$\begin{aligned}\varphi_{Y+Z}(u) &:= \mathbb{E}e^{iu(Y+Z)} = \mathbb{E}e^{iuY}e^{iuZ} \\ &= \mathbb{E}e^{iuY}\mathbb{E}e^{iuZ} = \varphi_Y(u)\varphi_Z(u).\end{aligned}$$

From this it easily follows that $\mathbb{E}e^{iu\sum_{n=1}^{k} Y_n} = \left(\mathbb{E}e^{iuY}\right)^k$.

Using this property may also easily calculate the characteristic function of a sum of two independent compound Poisson processes.

Characteristic functions of compound Poisson processes - cont.

Let

$$\mathbb{E}e^{iuX_t} = e^{\lambda t \mathbb{E}\left(e^{iuY}-1\right)} \text{ and } \mathbb{E}e^{iu\tilde{X}_t} = e^{\tilde{\lambda} t \mathbb{E}\left(e^{iu\tilde{Y}}-1\right)}$$

then

$$\begin{aligned}\mathbb{E}e^{iu\left(X_t+\tilde{X}_t\right)} &= e^{\lambda t \mathbb{E}\left(e^{iuY}-1\right)} e^{\tilde{\lambda} t \mathbb{E}\left(e^{iu\tilde{Y}}-1\right)} \\ &= e^{\left(\lambda+\tilde{\lambda}\right) t \mathbb{E}\left(\frac{\lambda}{\lambda+\tilde{\lambda}} e^{iuY} + \frac{\tilde{\lambda}}{\lambda+\tilde{\lambda}} e^{iu\tilde{Y}} - 1\right)}.\end{aligned}$$

Thus we see that the sum of two independent compound processes is again a compound Poisson process with the intensity $\lambda + \tilde{\lambda}$ and the distribution of jumps given as a mixture of the distributions of the variables Y and $\tilde{Y}$.

Characteristic exponents of compound Poisson processes

This may be even better seen when we write the expectations of e^{iuY} and $e^{iu\tilde{Y}}$ using the laws of the variables Y and $\tilde{Y}$. For a (Borel) set A let us define

$$\nu(A) := \mathbb{P}(Y \in A) \text{ and } \tilde{\nu}(A) := \mathbb{P}\left(\tilde{Y} \in A\right)$$

then

$$\mathbb{E}e^{iuY} - 1 = \int_{\mathbb{R}} \left\{e^{iuy} - 1\right\} \nu(\mathrm{d}y) \text{ and } \mathbb{E}e^{iu\tilde{Y}} - 1 = \int_{\mathbb{R}} \left\{e^{iuy} - 1\right\} \tilde{\nu}(\mathrm{d}y),$$

and

$$\begin{aligned}
&\lambda\mathbb{E}\left(e^{iuY} - 1\right) + \tilde{\lambda}\mathbb{E}\left(e^{iu\tilde{Y}} - 1\right) \\
&= \left(\lambda + \tilde{\lambda}\right) \int_{\mathbb{R}} \left\{e^{iuy} - 1\right\} \left\{\frac{\lambda}{\lambda + \tilde{\lambda}}\nu(\mathrm{d}y) + \frac{\tilde{\lambda}}{\lambda + \tilde{\lambda}}\tilde{\nu}(\mathrm{d}y)\right\}.
\end{aligned}$$

Characteristic functions of compound Poisson processes - cont.

As a result we see that any random variable whose characteristic function is given by

$$\mathbb{E}e^{iuX_1} = \exp\left(\int_{\mathbb{R}} \left\{e^{iuy} - 1\right\} \nu(\mathrm{d}y)\right),$$

where ν is a finite and non-negative (Borel) measure on $\mathbb{R}$ is a characteristic function of X_1 where X is a compound Poisson process with intensity $\Lambda := \nu(\mathbb{R} \setminus \{0\})$ such that

$$\mathbb{E}e^{iuX_t} = \exp\left(t \int_{\mathbb{R}} \left\{e^{iuy} - 1\right\} \nu(\mathrm{d}y)\right) = \exp\left(\Lambda t \int_{\mathbb{R}\setminus\{0\}} \left\{e^{iuy} - 1\right\} \frac{\nu(\mathrm{d}y)}{\Lambda}\right)$$

Moreover, the measure ν has the interpretation that $\nu(\mathrm{d}y)$ represents the intensity of jumps of size y.

Unfortunately (or fortunately ;-) it does not exhaust all possible forms of characteristic functions of Lévy processes without Brownian part.

A general form of the characteristic functions of real Lévy processes

Roughly speaking, it appears that it is possible to add small jumps, which appear with infinite intensity, but whose signs are opposite, which results in cancelation, or which are compensated by drift in opposite direction, and as a result we obtain finite sum.

A general form of the characteristic function of X_t is given by the formula

$$\mathbb{E}\exp(iuX_t) = \exp\left(iaut - \frac{1}{2}\sigma^2 u^2 t\right)$$
$$\times \exp\left(\nu(\mathbb{R}\setminus[-1,1])\, t \int_{\mathbb{R}\setminus(-1,1)} \left\{e^{iuy} - 1\right\} \frac{\nu(\mathrm{d}y)}{\nu(\mathbb{R}\setminus[-1,1])}\right)$$
$$\times \exp\left(t \int_{[-1,1]\setminus\{0\}} \left\{e^{iuy} - 1 - iuy\right\} \nu(\mathrm{d}y)\right).$$

Interpretation of a general form of the characteristic function of a real Lévy process

This formula makes sense under the assumption that

$$\int_{\mathbb{R}\setminus\{0\}} \min\left(y^2, 1\right) \nu\left(\mathrm{d}y\right) < +\infty.$$

- The first factor corresponds to the Brownian motion with drift $W_t = \sigma B_t + a \cdot t$, where B is a standard Brownian motion.
- the second factor corresponds to the compound Poisson process with the intensity $\nu\left(\mathbb{R} \setminus [-1, 1]\right)$ and the distribution of jums given by $\mathbf{1}_{\mathbb{R}\setminus[-1,1]} \frac{\nu(\mathrm{d}y)}{\nu(\mathbb{R}\setminus[-1,1])}$.
- the third factor corresponds to **compensated** sum of small jumps (with possibly infinite intensity).

Interpretation of the third factor of the characteristic function of a real Lévy process

Let us consider the sequence $1 = \varepsilon_0 > \varepsilon_1 > \ldots > 0$ such that $\lim_{n\to+\infty} \varepsilon_n = 0$ and for $n = 1, 2, \ldots$ and let us consider the factor

$$\exp\left(t\int_{A_n}\left\{e^{iuy}-1-iuy\right\}\nu(\mathrm{d}y)\right)$$
$$=\exp\left(\nu(A_n)\,t\int_{A_n}\left\{e^{iuy}-1\right\}\frac{\nu(\mathrm{d}y)}{\nu(A_n)}-iut\left(\int_{A_n}y\nu(\mathrm{d}y)\right)\right),$$

where we define $A_n := [-\varepsilon_{n-1}, -\varepsilon_n) \cup (\varepsilon_n, \varepsilon_{n-1}]$. This may be viewed as a characteristic function of a compensated compound Poisson process

$$X_t^{(n)} := \sum_{i=1}^{N_t^{(n)}} Y_i^{(n)} - \left(\int_{A_n} y\nu(\mathrm{d}y)\right)t,$$

where $N^{(n)}$ is a Poisson process with the intensity $\nu(A_n)$ and the distribution of jumps $Y^{(n)}$ is given by $\mathbf{1}_{A_n}\frac{\nu(\mathrm{d}y)}{\nu(A_n)}$.

$X^{(n)}$ is a square integrable martingale

Having the characteristic function of a variable X it is possible to calculate its moments:

$$\mathbb{E}X^k = (-i)^k \frac{\partial^k \mathbb{E}\exp(iuX)}{\partial u^k}|_{u=0}$$

(kth moment exist iff the above derivative exists). This follows from the expansion $\mathbb{E}\exp(iuX) = 1 + iu\mathbb{E}X + \frac{(iu)^2}{2!}\mathbb{E}X^2 + \ldots$. Having this we (easily ???) calculate

$$\mathbb{E}X_t^{(n)} = 0, \quad \mathbb{E}\left(X_t^{(n)}\right)^2 = t\int_{A_n} y^2 \nu(\mathrm{d}y).$$

From these equalities and the independence of increments of $X^{(n)}$ we infer that $X^{(n)}$ is a square integrable martingale.

Moreover, taking sequence of independent square integrable martingales $X^{(n)}$, $n = 1, 2, \ldots$ (on the same probability space) we get that the series $\sum_{n=1}^m X^{(n)}$ converges (as $m \to \infty$, in an appropriate topology) to a square integrable martingale.

Characteristic function of the sum $\sum_{n=1}^{\infty} X^{(n)}$

Since

$$\sum_{n=1}^{\infty} \int_{A_n} \left\{ e^{iuy} - 1 - iuy \right\} \nu(\mathrm{d}y) = \int_{[-1,1]\setminus\{0\}} \left\{ e^{iuy} - 1 - iuy \right\} \nu(\mathrm{d}y)$$

we get that the characteristic function of $\sum_{n=1}^{\infty} X_t^{(n)}$ is

$$\begin{aligned} &\prod_{n=1}^{\infty} \exp\left(t \int_{A_n} \left\{ e^{iuy} - 1 - iuy \right\} \nu(\mathrm{d}y) \right) \\ &= \exp\left(t \sum_{n=1}^{\infty} \int_{A_n} \left\{ e^{iuy} - 1 - iuy \right\} \nu(\mathrm{d}y) \right) \\ &= \exp\left(t \int_{[-1,1]\setminus\{0\}} \left\{ e^{iuy} - 1 - iuy \right\} \nu(\mathrm{d}y) \right). \end{aligned}$$

α-stable processes ($\alpha \in (0,2)$) - processes with infinite intensity of small jumps

An α-stable process with the stability parameter $\alpha \in (0,2)$ is a Lévy process with the following characteristic function

$$\mathbb{E}\exp(iuX_t) = \exp\left(iaut + t\int_{\mathbb{R}\setminus\{0\}}\left\{e^{iuy} - 1 - iuy\mathbf{1}_{[-1,1]}(y)\right\}\nu(\mathrm{d}y)\right),$$

where

$$\nu(\mathrm{d}y) = \frac{1}{|y|^{\alpha+1}}\left\{C_1\mathbf{1}_{(-\infty,0)}(y) + C_2\mathbf{1}_{(0,\infty)}(y)\right\}\mathrm{d}y$$

for some non-negative C_1 and C_2 such that $C_1 + C_2 > 0$.

Since $\nu([-1,1]) = +\infty$ we infer that these processes have infinite intensity of small jumps.

An α-stable process with the stability parameter $\alpha = 2$ is continuous - this is a Brownian motion with drift.

Strictly α-stable processes

- If $\alpha \in (0,1)$ then the characteristic function of an α-stable process X_t may be represented as

$$\mathbb{E}\exp(iuX_t) = \exp\left(ibut + t\int_{\mathbb{R}\setminus\{0\}} \left\{e^{iuy} - 1\right\}\nu(\mathrm{d}y)\right),$$

- if $\alpha \in (1,2)$ then the characteristic function of an α-stable process X_t may be represented as

$$\mathbb{E}\exp(iuX_t) = \exp\left(ibut + t\int_{\mathbb{R}\setminus\{0\}} \left\{e^{iuy} - 1 - iuy\right\}\nu(\mathrm{d}y)\right),$$

- the characteristic function of a 1-stable process X_t may be represented as

$$\mathbb{E}\exp(iuX_t) = \exp\left(ibut + \frac{2}{\pi}ict\beta u \ln|u| + ct|u|\right),$$

where $\beta \in [-1,1]$.

Strictly α-stable processes ($\alpha \in (0,2)$)- self-similar processes with infinite intensity of small jumps

If $\alpha \in (0,1) \cup (1,0)$ and $b = 0$ (or $\alpha = 1$ and $\beta = 0$) then the process is self-similar. More precisely, for any $A > 0$ one has (verify this using the characteristic function!)

$$\left(A^{-1/\alpha} X_{A \cdot s}\right)_{s \geq 0} =^{law} (X_s)_{s \geq 0} .$$

This may be easily verified using the characteristic functions (change of variable u).

The variable X_t is called (strictly) α-stable variable and it has the following property: if $X_t^{(1)}, \ldots, X_t^{(n)}$ are independent copies of X_t then

$$\frac{X_t^{(1)} + \ldots + X_t^{(n)}}{n^{1/\alpha}} =^{law} X_t.$$

This follows from the following easy calculation:

$$X_t^{(1)} + X_t^{(2)} + \ldots + X_t^{(n)}$$

$$=^{law} X_t^{(1)} + \left(X_{2t}^{(1)} - X_t^{(1)}\right) + \ldots + \left(X_{nt}^{(1)} - X_{(n-1)t}^{(1)}\right) = X_{nt}^{(1)} =^{law} n^{1/\alpha} X_t.$$

The jumps of a Lévy process - Poisson random measures

Let now X be a Lévy process. Let $t \in [0, +\infty)$ and A be a Borel subset of $\mathbb{R} \setminus \{0\}$. We consider the following **random** measure N on $[0, +\infty) \times \mathbb{R}$:

$$N([0, t] \times A) := \sum_{0<s\leq t} \mathbf{1}_A(\Delta X_s).$$

The measure $N([0, t] \times \mathrm{d}y)$ counts how many jumps of size y occured till the moment t.

Note that

$$\mathbb{E}N(t, A) = \int N(t, A) \, \mathrm{d}\mathbb{P}(\omega)$$

is a Borel measure on $\mathbb{R} \setminus \{0\}$. We will write $\mu(\cdot) = \mathbb{E}N(1, \cdot)$.

Definition

We say that the Borel set $A \subset \mathbb{R} \setminus \{0\}$ is bounded from below if $0 \notin \bar{A}$.

Poisson random measure related to a Lévy process - properties

The introduced random measures have the following properties.

- For each $t > 0$ and $\omega \in \Omega$, $N(t, \cdot)(\omega)$ is a counting measure on $\mathbb{R} \setminus \{0\}$;
- For each A bounded from below, $N(\cdot, A)$ is a Poisson process with the intensity $\mu(A)$;
- The **compensated** measure

$$\tilde{N}(t, \cdot)(\omega) := N(t, \cdot)(\omega) - t\mu(\cdot)$$

  is a martingale-valued measure, i.e. for any A bounded from below, $\tilde{N}(\cdot, A)$ is a martingale.
- Moreover, if $A \cap B = \emptyset$ then $N(\cdot, A)$ and $N(\cdot, B)$ are independent.

Poisson integrals

Let now $f : \mathbb{R} \to \mathbb{R}$ be a Borel-measurable function and $A \subset \mathbb{R}$ be a Borel set bounded from below. Now for $t \geq 0$ we define

$$\int_A f(x) N(t, \mathrm{d}x)(\omega) = \sum_{x \in A} f(x) N(t, \{x\})(\omega)$$
$$= \sum_{0 < s \leq t} f(\Delta X_s(\omega)) \mathbf{1}_A(\Delta X_s(\omega)).$$

We also define for $f \in L^1(A, \mu)$ the compensated process

$$\int_A f(x) \tilde{N}(t, \mathrm{d}x)(\omega) = \int_A f(x) N(t, \mathrm{d}x)(\omega) - t \int_A f(x) \mu(\mathrm{d}x)$$

which is a martingale.

Poisson integrals for $f(x) \equiv x$

For $f(x) \equiv x$ and $A^{(0)} = \mathbb{R} \setminus [-1, 1]$ we get that

$$\int_{A^{(0)}} xN(t, \mathrm{d}x)(\omega) = \sum_{0<s\leq t} \Delta X_s(\omega) \mathbf{1}_{A^{(0)}}(\Delta X_s(\omega))$$

is a compound Poisson process with the jumps > 1 occuring at exactly the same moments at the jumps of X. Moreover, the jumps of this process have exactly the same size as the jumps of of X. The difference

$$X - \int_{A^{(0)}} xN(t, \mathrm{d}x)$$

is a Lévy process with jumps smaller or equal 1. It can be shown that such a process has finite moments of **any order** (it takes some time for the process $|X|$ to grow and we control the rate of this growth).

Let us now consider the Lévy process defined as

$$\tilde{X}_t := X_t - \int_{A^{(0)}} xN(t, \mathrm{d}x) - t\mathbb{E}\left\{X_1 - \int_{A^{(0)}} xN(1, \mathrm{d}x)\right\}$$

This is a square-integrable, zero-mean martingale.

Proceeding to the limit

Next, for $n = 1, 2, \ldots$ we consider $A^{(n)} = (1/(n+1), 1/n]$ and the sequence of **independent**, zero-mean, square integrable martingales

$$M^{(n)} = \int_{A^{(n)}} x \tilde{N}(t, \mathrm{d}x).$$

For fixed (large) $T > 0$ and $N \in \mathbb{N}$ we have that $\tilde{X}_T - \sum_{n=1}^{N} M_T^{(n)}$ and $\sum_{n=1}^{N} M_T^{(n)}$ are independent, thus

$$\sum_{n=1}^{N} \mathbb{E}\left(M_T^{(n)}\right)^2 = \mathbb{E}\left(\tilde{X}_T\right)^2 - \sum_{n=1}^{N}\left(\tilde{X}_T - \sum_{n=1}^{N} M_T^{(n)}\right)^2 \le \mathbb{E}\left(\tilde{X}_T\right)^2.$$

From this we infer that $\sum_{n=1}^{N} M^{(n)}$ converges (at least on $[0, T]$) to some square integrable martingale M. The difference $\tilde{X} - M$ is a Lévy process without jumps (it is continuous), thus it is a Brownian motion.

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Thank you for your attention!