Irregular paths - between determinism and randomness

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Stochastic processes - mathematical tools for modelling the evolution of phenomena with uncertain outcomes

- **Stochastic processes** are flexible mathematical tools for modelling the evolution of phenomena with uncertain outcomes
- Examples of such phenomena are: stock prices, numbers of cases of some disease in a given area, the level of Nile river etc.
- In a most general setting, stochastic process is simple a collection of **random variables**

  $$X_t : \Omega \to E, \quad t \in T,$$

  where Ω is a probability space (equipped with a **probability function** $\mathbb{P} : \Omega \to [0, 1]$), E is the space of possible values of X_t and T is some set
- For given $\omega \in \Omega$ the **random function**

  $$T \ni t \mapsto X_t(\omega) \in E$$

  is called the **trajectory** of the process X

The standard Brownian motion

One of the most important stochastic processes is the d-dimensional **standard Brownian motion**. It is a process

$$B_t : \Omega \to \mathbb{R}^d, \quad t \in [0; +\infty)$$

(sometimes $t \in (-\infty; +\infty)$) such that

- for $s, t \in [0; +\infty)$, $B_t - B_s$ is a random variable with the **normal distribution** $\mathcal{N}(0, |t - s| Id)$
- B has independent increments, i.e. for $0 \leq s < t < u$, the increments

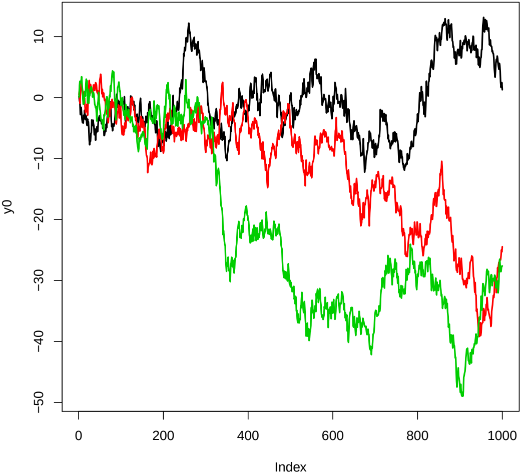
  $$B_u - B_t \text{ and } B_t - B_s$$

  are **independent**
- B has **continuous trajectories**

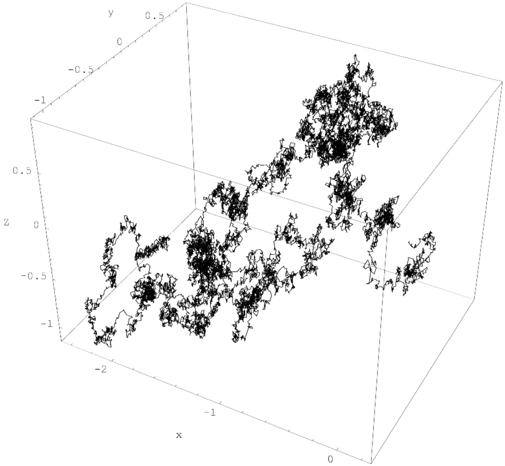
History of the Brownian motion

- The Brownian motion is named after a Scottish botanist Robert Brown who observed trajectories of a 2-dimensional Brownian motion. In 1827, while looking through a microscope at particles trapped in cavities inside pollen grains in water, he noted that the particles moved through the water; but he was not able to determine the mechanisms that caused this motion.
- Danish astronomer and actuary Thorvald Nicolai Thiele was the first person to give the mathematical model of the Brownian motion in 1880. Independently, it was done by French mathematician Louis Jean-Baptiste Alphonse Bachelier in his PhD thesis *The Theory of Speculation* (published in 1900), which discussed the use of Brownian motion to evaluate stock prices. It is historically the first paper to use advanced mathematics in the study of finance.
- The physical explanation of the movements of particles observed by Brown was given by Albert Einstein and by Polish physicist Marian (von) Smoluchowski.

Trajectories of a 1-dimensional standard Brownian motion



Trajectory of a 3-dimensional standard Brownian motion, source: Wikipedia



Infinite variation of a standard Brownian motion and a problem with the definition of an integral driven by this process

As we could see on the two previous slides, the trajectories of the Brownian motion are very irregular. They are only Hölder continuous with any exponent < 0.5, i.e. for any $T > 0$ and $\alpha < 0.5$

$$\sup_{0 \le s < t \le T} \frac{|B_t - B_s|}{|t - s|^\alpha} \text{ is almost surely finite.}$$

This makes it impossible to define for example the integral

$$\int_0^T B_s \mathrm{d} B_s$$

as the classical Riemann-Stieltjes integral, the Lebesgue-Stieltjes integral or the Young integral.

General theory of the Itô semimartingale integral

Systematically developed (since 1940, by Japanese and French schools) theory of the Itô stochastic integral led to the notion of a **semimartingale** and its **quadratic variation**.

Recipe for the stochastic integral $\int_0^T Y_- \mathrm{d}X$

Ingredients:

- a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$,
- a **càdlàg** semimartingale (with respect to the filtration $\mathbb{F}$) integrator $X_t \in \mathbb{R}^d$, $t \in [0, T]$,
- adapted (with respect to the filtration $\mathbb{F}$), càdlàg integrand process $Y_t \in \mathbb{R}^d$, $t \in [0, T]$.

Tools:

- quadratic variation $[X]$ (or - even better - predictable (left) version of the quadratic variation, $\langle X \rangle$),
- Itô's isomorphism or the theorem about the representation of continuous linear functionals on a Hilbert space.

Protter's approach

It is also possible (as in Philip Protter's book *Stochastic Integration and Differential Equations*) first to define integral for simple (adapted) integrand processes $Y_t = \sum_{i=1}^{+\infty} Y_i \mathbf{1}_{[\tau_{i-1},\tau_i)}(t)$, where $0 = \tau_0 \le \tau_1 \le \ldots$ are **stopping times** relative to the filtration $\mathbb{F}$, as

$$\int_0^T Y_- \mathrm{d}X = \sum_{i=1}^{+\infty} Y_i \cdot \left(X_{\min(\tau_i, T)} - X_{\min(\tau_{i-1}, T)} \right)$$

and then, to prove that biggest class of integrators for which this operation is continuous (in appropriate topology) is the class of semimartingales (with respect to the filtration $\mathbb{F}$). This is the famous **Dellacherie-Bichteler theorem**.

Quadratic variation

Whatever the approach is, the main tool used in the definition of the general stochastic integral is the quadratic variation $[X]_T$ of the semimartingale X_t, $t \in [0, T]$, defined as the following limit (in probability):

$$[X]_T =^{(P)} \lim_{n \to +\infty} X_0^2 + \sum_{i=1}^{N_n} \left(X_{t_i^n} - X_{t_{i-1}^n} \right)^2,$$

where $0 = t_0^n < t_1^n < \ldots < t_{N_n}^n = T$ is *any* sequence of (deterministic) partitions of the interval $[0, T]$ such that its **mesh** $\max_{i=1,\ldots,N_n} \left(t_i^n - t_{i-1}^n \right)$ tends to 0 as $n \to +\infty$.

Remark

For the d-dimensional standard Brownian motion almost surely we have

$$[B]_T = \langle B \rangle_T = dT.$$

Generalised Itô's formula (the Kunita-Watanabe formula)

For the semimartingale stochastic integral the following generalised Itô's formula (the Kunita-Watanabe formula) holds. If $F:\mathbb{R}^d\to\mathbb{R}$ is a function of the class C^2, then

$$
\begin{aligned}
F(X_T) = {} & F(X_0) + \sum_{i=1}^{d}\int_{0+}^{T}\frac{\partial F}{\partial x_i}(X_{s-})\,\mathrm{d}X_s^i \\
& + \frac{1}{2}\sum_{i,j=1}^{d}\int_{0+}^{T}\frac{\partial^2 F}{\partial x_i\partial x_j}(X_{s-})\,\mathrm{d}\left[X^i,X^j\right]_s^c \\
& + \sum_{0<s\le T}\left\{\Delta F(X_s) - \sum_{i=1}^{d}\frac{\partial F}{\partial x_i}(X_{s-})\,\Delta X_s^i\right\},
\end{aligned}
$$

where

$$
\left[X^i,X^j\right] = \frac{1}{4}\left(\left[X^i+X^j,X^i+X^j\right]-\left[X^i-X^j,X^i-X^j\right]\right)
$$

and $\left[X^i,X^j\right]^c$ is the continuous part of $\left[X^i,X^j\right]$.

Föllmer's approach

The constructions presented (or rather sketched) on the previous slides require a really big apparatus (filtrations, stopping times, (semi-)martingales, quadratic variation etc.). In his pioneer work from

1981, Hans Föllmer proved that the analog of the Kunita-Watanabe formula holds for any irregular, càdlàg path (thus **pathwise**), possessing appropriately defined quadratic variation.

Föllmer's approach, cont.

Definition

Let $x : [0, T] \to \mathbb{R}$ *be a càdlàg function and let* $\pi = (\pi^n)$, $n = 1, 2, \ldots,$ *be a sequence pf partitions of the interval* $[0, T]$,

$$\pi^n = \left\{0 = t_0^n < t_1^n \ldots < t_{N_n}^n = T\right\}$$

such that the mesh $\max_{i=1,2,\ldots,N_n} \left(t_i^n - t_{i-1}^n\right)$ *tends to* 0 *as* $n \to +\infty$. *Assume that the sequence of the measures*

$$\sum_{i=1}^{N_n} \left(x_{t_i^n} - x_{t_{i-1}^n}\right)^2 \delta_{\left\{t_{i-1}^n\right\}}$$

tends vaguely to a Radon measure ξ *on* $[0, T]$ *such that for any* $t \in [0, T]$, $\xi(\{t\}) = (\Delta x_t)^2$. *The quadratic variation of* x *with respect to the sequence of partitions* π *is defined as*

$$[x]_t := x_0^2 + \xi[0, t].$$

Föllmer's theorem

Theorem

If $x : [0, T] \to \mathbb{R}$ is càdlàg , has the quadratic variation along the sequence of partitions π and $f : \mathbb{R} \to \mathbb{R}$ is of the class C^2 then the following formula holds

$$
\begin{aligned}
f(x_T) = f(x_0) &+ \int_{0+}^{T} f'(x_{s-})\,\mathrm{d}x_s \\
&+ \frac{1}{2}\int_{0+}^{T} f''(x_{s-})\,\mathrm{d}[x]_s^c \\
&+ \sum_{0<s\le T}\left\{\Delta f(x_s) - f'(x_{s-})\,\Delta x_s\right\},
\end{aligned}
$$

where the integral $\int_{0+}^{T} f'(x_{s-})\,\mathrm{d}x_s$ is defined as the limit

$$
\int_{0+}^{T} f'(x_{s-})\,\mathrm{d}x_s = \lim_{n\to+\infty}\sum_{i=1}^{N_n} f\left(x_{t_{i-1}^n}\right)\left(t_i^n - t_{i-1}^n\right).
$$

Föllmer's theorem - remarks

- The proof of Föllmer's theorem is pretty simple, but the main problem is that *a priori* the integral $\int_{0+}^{T} f'(x_{s-})\,\mathrm{d}x_s$ and the quadratic variation $[x]$ depend on the sequence of partitions π.
- This should be emphasised in the notation, for example $(\pi)\int_{0+}^{T} f'(x_{s-})\,\mathrm{d}x_s$ and $(\pi)[x]$ since for two different sequences of partitions, π and ρ we may have

$$(\pi)\int_{0+}^{T} f'(x_{s-})\,\mathrm{d}x_s \neq (\rho)\int_{0+}^{T} f'(x_{s-})\,\mathrm{d}x_s$$

and

$$(\pi)[x] \neq (\rho)[x].$$

Föllmer's theorem - remarks cont.

- In fact, if x is a trajectory of a standard Brownian motion then it is almost surely continuous and almost surely it has infinite **(strong) 2-variation** defined as

$$V^2(x, [0, T]) := \sup_n \sup_{0 \le t_0 \le t_1 \ldots < t_n \le T} \sum_{i=1}^{n} \left(x_{t_i} - x_{t_{i-1}}\right)^2.$$

- A recent result of Mark Davis, Jan Obłój and Pietro Siorpaes states that for any such function and *any* non-decreasing function $C : [0, T] \to [0, +\infty)$ there exists a sequence of partitions π such that

$$(\pi)[x]_s = x_0^2 + C(s).$$

The quadratic variation vs. the truncated variation of semimartingales

From results of [L], [LM] and [LG] it follows that for any semimartingale $X_t,\ t \geq 0$, the following convergence holds almost surely:

$$\lim_{\varepsilon \to 0+} \varepsilon \cdot TV^{\varepsilon}(X, [0, T]) \to [X]^c_T, \tag{1}$$

where $TV^{\varepsilon}(X, [0, T])$ stands for the **truncated variation** defined as

$$TV^{\varepsilon}(X, [0, T]) := \sup_n \sup_{0 \leq t_0 \leq t_1 \ldots < t_n \leq T} \sum_{i=1}^{n} \max \left\{ \left(X_{t_i} - X_{t_{i-1}} \right) - \varepsilon, 0 \right\}.$$

Remark

Notice that the relation (1) may be treated as an alternative definition of the continuous part of the semimartingale quadratic variation and it **does not depend** *on any sequence of partitions*

The quadratic variation vs. the truncated variation of deterministic functions

Thus the following questions appear:

- For which *deterministic* càdlàg functions $x : [0, T] \to \mathbb{R}$ the continuous part of their quadratic variation $[x]^c_t$, $t \in [0, T]$ may be defined as the limit

$$\lim_{\varepsilon \to 0+} \varepsilon \cdot \mathrm{TV}^{\varepsilon}(x, [0, t])?$$

- Is it possible to state for such functions an analog of Föllmer's theorem?

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Thank you for your attention