Integrals driven by irregular signals in Banach spaces and rate-independent characteristics of their irregularity

Rafał M. Łochowski

Warsaw School of Economics

Fractality and Fractionality 2016

p-variation as a measure of irregularity

One of the most popular measures of irregularity of a deterministic or a random path attaining values in a Banach space E, $X : [0, T] \to E$, is p-variation ($p > 0$).
One should distinct the **equidistant p-variation**, which is defined as the limit (if it exists)

$$EqdstV^p (X, [0, T]) := \lim_{n \to +\infty} \sum_{i=1}^{n} \left\| X \left(T \cdot \frac{i}{n} \right) - X \left(T \cdot \frac{i-1}{n} \right) \right\|^p$$

p-variation as a measure of irregularity

One of the most popular measures of irregularity of a deterministic or a random path attaining values in a Banach space E, $X : [0, T] \to E$, is p-variation ($p > 0$).
One should distinct the **equidistant p-variation**, which is defined as the limit (if it exists)

$$EqdstV^p\left(X, [0, T]\right) := \lim_{n \to +\infty} \sum_{i=1}^{n} \left\| X\left(T \cdot \frac{i}{n}\right) - X\left(T \cdot \frac{i-1}{n}\right) \right\|^p$$

and the **strong p-variation**, which is defined as

$$V^p\left(X, [0, T]\right) := \sup_{n=1,2,\ldots} \sup_{0 \le t_0 < t_1 < \ldots < t_n \le T} \sum_{i=1}^{n} \left\| X\left(t_n\right) - X\left(t_{n-1}\right) \right\|^p .$$

p-variation as a measure of irregularity

One of the most popular measures of irregularity of a deterministic or a random path attaining values in a Banach space E, $X : [0, T] \to E$, is p-variation ($p > 0$).

One should distinct the **equidistant p-variation**, which is defined as the limit (if it exists)

$$EqdstV^p(X, [0, T]) := \lim_{n \to +\infty} \sum_{i=1}^{n} \left\| X\left(T \cdot \frac{i}{n}\right) - X\left(T \cdot \frac{i-1}{n}\right) \right\|^p$$

and the **strong p-variation**, which is defined as

$$V^p(X, [0, T]) := \sup_{n=1,2,\ldots} \sup_{0 \le t_0 < t_1 < \ldots < t_n \le T} \sum_{i=1}^{n} \| X(t_n) - X(t_{n-1}) \|^p .$$

The strong 1-variation is also called the **total variation** and it is important in the theory of the Stieltjes integral.

p-variation of a fBm

It is well known that the equidistant p-variation of a real-valued fBm B^H with the Hurst parameter $H \in (0,1)$ equals (a.s.)

- $EqdstV^p\left(B^H, [0, T]\right) = +\infty$ for $p < 1/H$,
- $EqdstV^p\left(B^H, [0, T]\right) = T \cdot \mathbb{E}\left|B_1^H\right|^{1/H}$ for $p = 1/H$,
- $EqdstV^p\left(B^H, [0, T]\right) = 0$ for $p > 1/H$,

see [Guerra, Nualart 2005].

p-variation of a fBm

It is well known that the equidistant p-variation of a real-valued fBm B^H with the Hurst parameter $H \in (0,1)$ equals (a.s.)

- $EqdstV^p\left(B^H, [0, T]\right) = +\infty$ for $p < 1/H$,
- $EqdstV^p\left(B^H, [0, T]\right) = T \cdot \mathbb{E}\left|B_1^H\right|^{1/H}$ for $p = 1/H$,
- $EqdstV^p\left(B^H, [0, T]\right) = 0$ for $p > 1/H$,

see [Guerra, Nualart 2005].
However, for the strong p-variation the picture is different. It is always a finite random quantity for $p > 1/H$ and

$$V^p\left(B^H, [0, T]\right) = +\infty \text{ for } p \leq 1/H \text{ (a.s.)}$$

see [Pratelli 2006], [Ciesielski, Kerkyacharian, Roynette 1993].

Rate dependent vs. rate independent measures of irregularity

Definition

We will say that the measure of irregularity $MIrr\,(X, [T_1, T_2])$ *of a path* $X : [T_1, T_2] \to E$ *is* **rate-independent** *if for any continuous and no-decreasing time change* $\tau : [T_1, T_2] \to [S_1, S_2]$ *such that* $\tau(T_1) = S_1$ *and* $\tau(T_2) = S_2$ *one has*

$$MIrr\,(X, [T_1, T_2]) = MIrr\,(Y, [S_1, S_2])\,,$$

where $Y : [S_1, S_2] \to E,\ Y(s) = X(\inf\{t : \tau(t) = s\})$.
If $MIrr\,(X, [T_1, T_2])$ *is not rate independent, we will say that it is* **rate-dependent**.

Rate dependent vs. rate independent measures of irregularity

Definition

We will say that the measure of irregularity $MIrr\left(X,[T_1,T_2]\right)$ *of a path* $X:[T_1,T_2]\to E$ *is* **rate-independent** *if for any continuous and no-decreasing time change* $\tau:[T_1,T_2]\to[S_1,S_2]$ *such that* $\tau(T_1)=S_1$ *and* $\tau(T_2)=S_2$ *one has*

$$MIrr\left(X,[T_1,T_2]\right)=MIrr\left(Y,[S_1,S_2]\right),$$

where $Y:[S_1,S_2]\to E,\ Y(s)=X(\inf\{t:\tau(t)=s\})$.
If $MIrr\left(X,[T_1,T_2]\right)$ *is not rate independent, we will say that it is* **rate-dependent**.

The equidistant p-variation is rate-dependent, while the strong p-variation is rate independent measure of irregularity.

Other examples of rate dependent and rate independent measures of irregularity

Another rate-dependent measure of irregularity is the **Hölder continuity** or the **Besov space regularity**, which is more precise than the Hölder continuity.

Other examples of rate dependent and rate independent measures of irregularity

Another rate-dependent measure of irregularity is the **Hölder continuity** or the **Besov space regularity**, which is more precise than the Hölder continuity.

Another, rate-independent measure of irregularity is (strong) ψ**-variation**, defined for a non-decreasing $\psi : [0, +\infty) \to [0, +\infty)$ similarly as the (strong) p-variation:

$$V^{\psi}(X, [0, T]) := \sup_{n=1,2,\ldots} \sup_{0 \le t_0 < t_1 < \ldots < t_n \le T} \sum_{i=1}^{n} \psi(\|X(t_n) - X(t_{n-1})\|).$$

Other examples of rate dependent and rate independent measures of irregularity

Another rate-dependent measure of irregularity is the **Hölder continuity** or the **Besov space regularity**, which is more precise than the Hölder continuity.

Another, rate-independent measure of irregularity is (strong) ψ-**variation**, defined for a non-decreasing $\psi : [0, +\infty) \to [0, +\infty)$ similarly as the (strong) p-variation:

$$V^{\psi}(X, [0, T]) := \sup_{n=1,2,\ldots} \sup_{0 \le t_0 < t_1 < \ldots < t_n \le T} \sum_{i=1}^{n} \psi\left(\left\|X(t_n) - X(t_{n-1})\right\|\right).$$

The greater is the order of ψ at 0 such that $V^{\psi}(X, [0, T]) < +\infty$, the higher is the degree of regularity of the path X.

Optimal result on the strong ψ-variation of a fBm

For the real fBm it is well known [Dudley, Norvaisa 2011] that for

$$\psi^{1/H}(x) = \frac{x^{1/H}}{\sqrt{\ln \ln (\max (1/x, 3))}^{1/H}}$$

and any $T > 0$, one has

$$V^{\psi^{1/H}}\left(B^H, [0, T]\right) < +\infty \text{ (a.s.)},$$

Optimal result on the strong ψ-variation of a fBm

For the real fBm it is well known [Dudley, Norvaisa 2011] that for

$$\psi^{1/H}(x) = \frac{x^{1/H}}{\sqrt{\ln\ln\left(\max\left(1/x, 3\right)\right)}^{1/H}}$$

and any $T > 0$, one has

$$V^{\psi^{1/H}}\left(B^H, [0, T]\right) < +\infty \text{ (a.s.)},$$

while for any $\varphi : [0, +\infty) \to [0, +\infty)$ such that $\lim_{x\to 0+} \frac{\varphi(x)}{\psi^{1/H}(x)} = +\infty$ one has

$$V^{\varphi}\left(B^H, [0, T]\right) = +\infty \text{ (a.s.)}.$$

Optimal result on the strong ψ-variation of a fBm

For the real fBm it is well known [Dudley, Norvaisa 2011] that for

$$\psi^{1/H}(x) = \frac{x^{1/H}}{\sqrt{\ln \ln (\max (1/x, 3))}^{1/H}}$$

and any $T > 0$, one has

$$V^{\psi^{1/H}} \left(B^H, [0, T] \right) < +\infty \text{ (a.s.)},$$

while for any $\varphi : [0, +\infty) \to [0, +\infty)$ such that $\lim_{x \to 0+} \frac{\varphi(x)}{\psi^{1/H}(x)} = +\infty$ one has

$$V^{\varphi} \left(B^H, [0, T] \right) = +\infty \text{ (a.s.)}.$$

The same holds for the Banach space-valued fBm, when the Banach-space valued fBm $G^H : [0, T] \to E$ with the Hurst parameter H is defined as a process such that for any $e^* \in E^*$, $< G^H, e^* >$ is a real fBm B^H multiplied by some constant.

The truncated variation - variational definition

From the results stated on the previous slide it immediately follows that the total variation (the strong 1-variation) of a Banach-space-valued fBm B^H with any Hurst parameter $H \in (0, 1)$ is a.s. infinite.
However, since the fBm is a continuous process, its paths may be uniformly approximated (with arbitrary accuracy) by paths with finite total variation.

The truncated variation - variational definition

From the results stated on the previous slide it immediately follows that the total variation (the strong 1-variation) of a Banach-space-valued fBm B^H with any Hurst parameter $H \in (0,1)$ is a.s. infinite.
However, since the fBm is a continuous process, its paths may be uniformly approximated (with arbitrary accuracy) by paths with finite total variation.
Let us fix $c > 0$ and consider a random quantity defined as

$$\mathrm{InfTV}^c\left(B^H, [0,T]\right) := \inf\left\{V^1\left(Y, [0,T]\right) : \|Y - B^H\|_{[0,T],\infty} \le c/2\right\},$$

where $\|Y - B^H\|_{[0,T],\infty} := \sup_{t\in[0,T]} \|Y_t - B_t^H\|$ is the supremum norm and the infimum is over **all** stochastic processes Y.

The truncated variation - semi-explicit definition

It is possible to prove that for the just defined quantity $\mathrm{InfTV}^{c}(X,[0,T])$ is bounded by the following semi-explicit quantity:

$$\mathrm{TV}^{c}(X,[0,T]) := \sup_{n} \sup_{0\leq t_0<t_1<\ldots<t_n\leq T} \sum_{i=1}^{n} \max\left(\left\| X\left(t_n\right) - X\left(t_{n-1}\right)\right\| - c, 0\right). \tag{1}$$

The truncated variation - semi-explicit definition

It is possible to prove that for the just defined quantity $\mathrm{InfTV}^c(X, [0, T])$ is bounded by the following semi-explicit quantity:

$$\mathrm{TV}^c(X, [0, T]) := \sup_n \sup_{0 \le t_0 < t_1 < \ldots < t_n \le T} \sum_{i=1}^{n} \max\left(\left\| X\left(t_n\right) - X\left(t_{n-1}\right)\right\| - c, 0\right). \quad (1)$$

More precisely, for any **regulated** (i.e. with right and left limits) path X one has the following bounds [Łochowski 2016]

$$\mathrm{TV}^c(X, [0, T]) \le \mathrm{InfTV}^c(X, [0, T]) \le 2 \cdot \mathrm{TV}^{c/4}(X, [0, T]). \quad (2)$$

We will call the quantity defined by the formula (1) **truncated variation**.

Asymptotics of the truncated variation as a rate-independent measure of irregularity

- From Eq. (2) it follows that the asymptotics of the truncated variation $\mathrm{TV}^c(X, [0, T])$, as $c \to 0+$, may be viewed as another rate-independent measure of irregularity of the path X.

Asymptotics of the truncated variation as a rate-independent measure of irregularity

- From Eq. (2) it follows that the asymptotics of the truncated variation $\mathrm{TV}^c(X,[0,T])$, as $c \to 0+$, may be viewed as another rate-independent measure of irregularity of the path X.
- The natural interpretation of this asymptotics is the following: what is the order of the total variation of any path Y uniformly approximating the path X with accuracy c as $c \to 0+$?

Asymptotics of the truncated variation as a rate-independent measure of irregularity

- From Eq. (2) it follows that the asymptotics of the truncated variation $\mathrm{TV}^c(X,[0,T])$, as $c \to 0+$, may be viewed as another rate-independent measure of irregularity of the path X.
- The natural interpretation of this asymptotics is the following: what is the order of the total variation of any path Y uniformly approximating the path X with accuracy c as $c \to 0+$?
- There exists interesting relationships between this asymptotics of the truncated variation and the ψ-variation. For example, from the inequality $c^{p-1}\max(|x|-c,0) \le |x|^p$ it immediately follows that

$$\sup_{c>0} c^{p-1}\mathrm{TV}^c(X,[0,T]) \le V^p(X,[0,T]).$$

Thus, for any path with finite p-variation, where $p \ge 1$, the truncated variation grows no faster than c^{p-1} as $c \to 0+$.

Relationships between the truncated variation and the strong p-variation

The property that

$$\sup_{c>0} c^{p-1} \mathrm{TV}^c(X, [0, T]) < +\infty \tag{3}$$

appears to be weaker than the property that $V^p(X, [0, T]) < +\infty$ and the paths of a real fBm B^H provide (a.s.) an example.

Relationships between the truncated variation and the strong p-variation

The property that

$$\sup_{c>0} c^{p-1} TV^c(X, [0, T]) < +\infty \tag{3}$$

appears to be weaker than the property that $V^p(X, [0, T]) < +\infty$ and the paths of a real fBm B^H provide (a.s.) an example.
From the scaling property and stationarity of the increments of fBm it follows that they have the following property:

$$\sup_{c>0} c^{1/H-1} TV^c\left(B^H, [0, T]\right) < +\infty \text{ (a.s.)} \tag{4}$$

but (as it was already mentioned)

$$V^{1/H}\left(B^H, [0, T]\right) = +\infty \text{ (a.s.)}.$$

Relationships between the truncated variation and the strong ψ-variation, open questions

On the other hand, from property (3) it follows that for any $\gamma > 1$

$$V^{\psi_\gamma^p}(X, [0, T]) < +\infty$$

where

$$\psi_\gamma^p(x) = \frac{x^p}{\ln(2 + 1/x)(\ln\ln(3 + 1/x))^\gamma},$$

see [Łochowski 2016].

Relationships between the truncated variation and the strong ψ-variation, open questions

On the other hand, from property (3) it follows that for any $\gamma > 1$

$$V^{\psi_\gamma^p}(X, [0, T]) < +\infty$$

where

$$\psi_\gamma^p(x) = \frac{x^p}{\ln(2 + 1/x)(\ln \ln(3 + 1/x))^\gamma},$$

see [Łochowski 2016].
As a result, we get that $V^{\psi_\gamma^{1/H}}\left(B^H, [0, T]\right) < +\infty$. Unfortunately, $\psi_\gamma^{1/H}$ has smaller order as $c \to 0+$ than the optimal $\psi^{1/H}$.

Relationships between the truncated variation and the strong ψ-variation, open questions

On the other hand, from property (3) it follows that for any $\gamma > 1$

$$V^{\psi_\gamma^p}(X, [0, T]) < +\infty$$

where

$$\psi_\gamma^p(x) = \frac{x^p}{\ln(2 + 1/x)(\ln\ln(3 + 1/x))^\gamma},$$

see [Łochowski 2016].

As a result, we get that $V^{\psi_\gamma^{1/H}}(B^H, [0, T]) < +\infty$. Unfortunately, $\psi_\gamma^{1/H}$ has smaller order as $c \to 0+$ than the optimal $\psi^{1/H}$.

The following open question arises: what is highest order of non-decreasing $\psi : [0, +\infty) \to [0, +\infty)$ such that $V^\psi(X, [0, T]) < +\infty$ for any X satisfying (3).

Relationships between the truncated variation and the strong ψ-variation, open questions

On the other hand, from property (3) it follows that for any $\gamma > 1$

$$V^{\psi_\gamma^p}(X, [0, T]) < +\infty$$

where

$$\psi_\gamma^p(x) = \frac{x^p}{\ln(2 + 1/x)(\ln\ln(3 + 1/x))^\gamma},$$

see [Łochowski 2016].

As a result, we get that $V^{\psi_\gamma^{1/H}}(B^H, [0, T]) < +\infty$. Unfortunately, $\psi_\gamma^{1/H}$ has smaller order as $c \to 0+$ than the optimal $\psi^{1/H}$.

The following open question arises: what is highest order of non-decreasing $\psi : [0, +\infty) \to [0, +\infty)$ such that $V^\psi(X, [0, T]) < +\infty$ for any X satisfying (3).

The relationship with the Besov space regularity is also an open problem. For a partial result see [Rosenbaum 2009].

Some norms

It appears that the quantity

$$\|X\|_{p-TV,[0,T]} := \left(\sup_{c>0} c^{1/H-1}\mathrm{TV}^c\left(B^H,[0,T]\right)\right)^{1/p}$$

is a norm and we have the estimate

$$\|X\|_{p-TV,[0,T]} \le \|X\|_{p-var,[0,T]},$$

where

$$\|X\|_{p-var,[0,T]} := \left(V^p\left(X,[0,T]\right)\right)^{1/p},$$

but the norms $\|\cdot\|_{p-TV,[0,T]}$ and $\|\cdot\|_{p-var,[0,T]}$ are not equivalent.

Some application - pathwise integrals with respect to irregular paths

We have the following stronger version of the Loéve-Young inequality (valid also for Banach space-valued paths).

Theorem

If $p, q > 1$, $1/p + 1/q > 1$ and

$$\|X\|_{p-TV,[0,T]} < +\infty, \quad \|Y\|_{q-TV,[0,T]} < +\infty$$

then the Riemann-Stieltjes integral $\int_0^T X \mathrm{d} Y$ exists and we have the following inequality

$$\left\| \int_0^T X \mathrm{d} Y - X_0 \left[Y_T - Y_0 \right] \right\| \leq C_{p,q} \|X\|_{p-TV,[0,T]} \|Y\|_{q-TV,[0,T]}$$

for some constant $C_{p,q}$ depending only on p and q.

Some application - pathwise integrals with respect to irregular paths

Moreover, for the indefinite integral $\int_0^{\cdot} X \mathrm{d} Y$ defined as $t \mapsto \int_0^t X \mathrm{d} Y$ one has

$$\| \int_0^{\cdot} X \mathrm{d} Y \|_{p-TV,[0,T]} \leq D_{p,q} \| X \|_{p-TV,[0,T]} \| Y \|_{q-TV,[0,T]}$$

for some constant $D_{p,q}$ depending only on p and q.

Some application - pathwise integrals with respect to irregular paths

Moreover, for the indefinite integral $\int_0^{\cdot} X \mathrm{d} Y$ defined as $t \mapsto \int_0^t X \mathrm{d} Y$ one has

$$\| \int_0^{\cdot} X \mathrm{d} Y \|_{p-TV,[0,T]} \leq D_{p,q} \|X\|_{p-TV,[0,T]} \|Y\|_{q-TV,[0,T]}$$

for some constant $D_{p,q}$ depending only on p and q.

Since $\left\|B^H\right\|_{1/H-TV,[0,T]} < +\infty$ but $\left\|B^H\right\|_{1/H-var,[0,T]} = +\infty$ this estimate applied to the fBm gives stronger result than the classical pathwise Loéve-Young estimate.

References

[Ciesielski, Kerkyacharian, Roynette 1993] *Quelques espaces fonctionnelles associs a des processus Gaussiens.* Studia Math., 107(2):171–204.
[Dudley, Norvaisa 2010] *Concrete Functional Calculus.* Springer Monographs in Mathematics, Springer, New York (2010) 683 pp.
[Guerra, Nualart 2005] *The 1=H-variation of the divergence integral with respect to the fractional Brownian motion for H ¿ 1/2 and fractional Bessel processes.* Stoch. Process. Appl. 115(1), 91115.
[Łochowski 2016] *Riemann-Stieltjes integrals driven by irregular signals in Banach spaces and rate-independent characteristics of their irregularity.* arXiv:1602.02269.
[Pratelli 2006] *A remark on the 1/H-variation of the Fractional Brownian Motion.* In Séminaire de Probabilités XLIII, volume 215 of Lecture Notes in Math., pp. 215–219. Springer, Berlin, 2006.
[Rosenbaum 2009] *First order p-variations and Besov spaces.* Statist. Probab. Lett. 79 (1), 5562.

Thank you for your attention