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FULL COOPERATION APPLIED TO ENVIRONMENTAL IMPROVEMENTS

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Abstract. We analyse the case of certificates of environmental improvements and full cooperation of two identical agents. We model pollution levels as geometric Brownian motions with quadratic costs of improvements. Our main result is the construction of the optimal improvements strategy in the case of separate actions, collusive actions and fusion. In certain range of the model parameters, the fusion solution generates lower pollution levels than separate and collusive actions.

1. Introduction. Among many proposals to halt environmental destruction, our goes through the emission of certificates of improvements that stimulate cooperation and free transfer of technologies. We use soft application of Principal-Agent methodology. The choice of certificates issued by Principal (in other words Nature), represented by a fi-

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nancial institution, is of secondary importance for it is impossible to establish proper optimality of it. We simply choose it as Fund-square of pollution levels, and show in which cases full cooperation (fusion) reflected by the emission of certificates that embraces two domains will produce better results than separate actions.

We set $T=1$ as time horizon. We specifically analyse three cases:

1. Separate actions: different certificates are issued for each agent.
2. Fusion: there is one certificate for two domains and it could lead to the free transfer of technologies because, in practice, two agents act jointly as a one.
3. Collusive actions: there is one certificate that embraces two domains, but although each agent has full information about pollution levels in other's domain at any time $0 \le t \le 1$, she/he can make improvements in her or his domain only.

Let $X_i(t)$, $i=1,2$, $t \in [0;1]$ denote the pollution levels in the ith domain at the time t. In the first case the ith agent minimizes

$$E\left[X_i^2(1) + \text{cost of improvements}\right], \ i=1,2.$$

In the second and third cases the agents minimize

$$E\left[\left(X_1(1)+X_2(1)\right)^2 + \text{cost of improvements}\right].$$

In all cases X_i, $i=1,2$, are driven by (possibly dependent) standard Brownian motions.

We will apply the dynamic programing approach, in its martingale form, and use quadratic cost function to stimulate cooperation. Certificates of improvements would pay $S_i - X_i^2(1)$ or $S-(X_1(1)+X_2(1))^2$. (We assume here that funds S_i, S, would be sufficient in the sense that these amounts remain nonnegative.)

2. The different models.

2.1. Separate certificates (actions). This case is the easiest one. The two processes X_i, $i=1,2$, are driven by (possibly dependent) standard Brownian motions. More precisely, for $i=1,2$ we assume the following dynamics of the pollution level X_i in the ith domain:

$$\mathrm{d}X_i(t) = \alpha X_i(t)dt + \beta X_i(t)\mathrm{d}W_i(t) - Au_i(t)\mathrm{d}t, \tag{1}$$

with $X_i(0)=x_i>0$ and $u_i(t)\ge 0$, $0\le t\le 1$, representing improvements in the ith domain. Here W_1, W_2 are Brownian motions; $\alpha, A \ge 0$, $\beta \ne 0$ are model parameters. The filtration that we consider is the natural filtration of the two Brownian motions. To avoid technical difficulties we will consider only adapted controls u_i, $i=1,2$, such that for $i=1,2$, $E\int_0^1 u_i^{16}(s)\mathrm{d}s < +\infty$ (this assumption will be used in the sequel).

Furthermore, we assume quadratic costs of improvements i.e., $\int_0^1 u_i^2(s)\mathrm{d}s$, $i=1,2$.

The ith agent minimizes

$$E\left(X_i^2(1) + \int_0^1 u_i^2(s)\mathrm{d}s\right). \tag{2}$$

We will show that there exists a smooth deterministic function $y(t)$, $0 \le t \le 1$, satisfying $y(1) = 1$, and an adapted process $u_i^* \geq 0$ such that

$$Z_i^*(t) := y(t)X_i^2(t) + \int_0^t u_i^{*2}(s)\mathrm{d}s$$

is a martingale, and for any other admissible control u_i,

$$Z_i(t) := y(t)X_i^2(t) + \int_0^t u_i^2(s)\mathrm{d}s$$

is a submartingale. Therefore,

$$E\left(X_i^2(1) + \int_0^1 u_i^{*2}(s)\mathrm{d}s\right) = E\left(y(1)X_i^2(1) + \int_0^1 u_i^{*2}(s)\mathrm{d}s\right) = y(0)X_i^2(0)$$
$$\leq E\left(y(1)X_i^2(1) + \int_0^1 u_i^2(s)\mathrm{d}s\right) = E\left(X_i^2(1) + \int_0^1 u_i^2(s)\mathrm{d}s\right).$$

Thus u_i^* solves the optimization problem, moreover,

$$E\left(X_i^2(1) + \int_0^1 u_i^{*2}(s)\mathrm{d}s\right) = y(0)X_i^2(0) = y(0)x_i^2.$$

To prove this, first we will justify that under our choice of control processes the process Z_i has a true martingale part. This holds as soon as $yX_i^2\mathrm{d}W_i$ corresponds to a true martingale, which, by the Itô isometry, holds when $E\int_0^1 X_i^4(t)\mathrm{d}t$ is finite. To estimate $E\int_0^1 X_i^4(t)\mathrm{d}t$ notice that the solution of (1) is given by $X_i = U_iV_i$, where

$$U_i(t) = \mathrm{e}^{\left(\alpha - \frac{1}{2}\beta^2\right)t + \beta W_i(t)}$$

and

$$V_i(t) = x_i - A\int_0^t U_i^{-1}(s)\, u_i(s)\,\mathrm{d}s.$$

By the elementary inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ we estimate

$$EX_i^4 = EU_i^4V_i^4 \leq \frac{1}{2}EU_i^8 + \frac{1}{2}EV_i^8. \tag{3}$$

It is obvious that $EU_i^8(t) = \exp\left(28\beta^2 t + 8\alpha t\right)$ is a continuous and bounded function for $t \in [0;1]$. Next, using the elementary inequalities $(a+b)^8 \leq 2^7a^8 + 2^7b^8$ and $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ we estimate

$$\begin{aligned} EV_i^8(t) &= E\left(x_i - A\int_0^t U_i^{-1}(s)\, u_i(s)\,\mathrm{d}s\right)^8 \\ &\leq 2^7x_i^8 + 2^7A^8E\left(\int_0^t U_i^{-1}(s)\, u_i(s)\,\mathrm{d}s\right)^8 \\ &\leq 2^7x_i^8 + 2^7A^8E\left(\frac{1}{2}\int_0^t U_i^{-2}(s)\,\mathrm{d}s + \frac{1}{2}\int_0^t u_i^2(s)\,\mathrm{d}s\right)^8 \\ &\leq 2^7x_i^8 + 2^6A^8E\left(\int_0^t U_i^{-2}(s)\,\mathrm{d}s\right)^8 + 2^6A^8E\left(\int_0^t u_i^2(s)\,\mathrm{d}s\right)^8. \end{aligned} \tag{4}$$

Further, using the Hölder inequality we prove that

$$E\left(\int_0^t U_i^{-2}(s)\,\mathrm{d}s\right)^8 \le \left(\int_0^t 1\mathrm{d}s\right)^7 E\int_0^t U_i^{-16}(s)\,\mathrm{d}s$$
$$= t^7\frac{\exp\left(136\beta^2 t - 16\alpha t\right)-1}{136\beta^2 t - 16\alpha t}, \tag{5}$$

hence $E\left(\int_0^t U_i^{-2}(s)\,\mathrm{d}s\right)^8$ is bounded for $t\in[0;1]$. In a similar way and by the assumption on u_i, for $t\in[0;1]$ we have

$$E\left(\int_0^t u_i^2(s)\,\mathrm{d}s\right)^8 \le t^7 E\int_0^t u_i^{16}(s)\,\mathrm{d}s < +\infty. \tag{6}$$

From (3)-(6) we get that $\sup_{t\in[0;1]} EX_i^4(t)$ is finite. Hence $E\int_0^1 X_i^4(t)\mathrm{d}t = \int_0^1 EX_i^4(t)\mathrm{d}t < +\infty$, thus Z_i indeed has a true martingale part.

Now let y be the solution of the following ordinary differential equation:

$$y'(t) = A^2y^2(t) - (2\alpha+\beta^2)y(t). \tag{7}$$

The equation (7) has a unique positive solution in $(-\infty;1]$ satisfying $y(1)=1$, namely

$$y(t) = \frac{1}{\frac{A^2}{B}\left(1-\mathrm{e}^{B(t-1)}\right)+\mathrm{e}^{B(t-1)}} > 0, \tag{8}$$

where $B = 2\alpha+\beta^2>0$. Now we easily calculate the drift part of Z_i, which reads

$$\underbrace{\left((2\alpha+\beta^2)y+y'\right)X_i^2}_{I} + \underbrace{u_i^2 - 2AyX_iu_i}_{II}. \tag{9}$$

This may by estimated by

$$\begin{aligned}
&\left((2\alpha+\beta^2)y+y'\right)X_i^2 + u_i^2 - 2AX_iy\cdot u_i\\
&= \left((2\alpha+\beta^2)y+y'\right)X_i^2 + (u_i - AX_iy)^2 - A^2X_i^2y^2\\
&\ge \left((2\alpha+\beta^2)y+y'\right)X_i^2 - A^2X_i^2y^2\\
&= X_i^2\left((2\alpha+\beta^2)y+y'-A^2y^2\right)\\
&= 0,
\end{aligned} \tag{10}$$

where the last equality follows from (7). Thus Z_i is a submartingale.

Now let X_i satisfy $X_i(0)=x_i$ and (1) with

$$u_i = u_i^* = AX_iy. \tag{11}$$

Substituting this into (1), we get that

$$\mathrm{d}X_i(t) = \left(\alpha - A^2y(t)\right)X_i(t)\mathrm{d}t + \beta X_i(t)\mathrm{d}W_i(t),$$

thus X_i is a generalized geometric Brownian motion and

$$u_i^*(t) > 0 \text{ for } t\in(-\infty;1]. \tag{12}$$

We also easily check that $\sup_{t\in[0;1]} E\left(u_i^*(t)\right)^{16} < +\infty$ thus u_i^* is admissible. Finally, notice that substituting (11) into (9) we get that the drift of Z_i^* equals 0 and Z_i^* is a martingale.

Remark 2.1. Solution of the optimisation problem we have just obtained may be applied for any nonnegative parameter A and any real parameters α, β. Namely, for any $A \geq 0$, $\alpha, \beta \in \mathbb{R}$ such that $B = 2\alpha + \beta^2 \neq 0$ the solution of (7) with the boundary condition $y(1) = 1$ is given by (8) and (12) holds. In the case $B = 2\alpha + \beta^2 = 0$ we obtain $y(t) = \left(1 + A^2 (1-t)\right)^{-1}$ and again (12) holds.

2.2. Fusion. In the fusion case each agent may act in both domains (in fact they act as a single agent) and the dynamics of X_1, X_2 is given by

$$\mathrm{d}X_1(t) = \alpha X_1(t)\mathrm{d}t + \beta X_1(t)\mathrm{d}W_1(t) - \left(Au_1(t) + Au_2(t)\right) I_{\{X_1(t)>0\}}\mathrm{d}t, \tag{13}$$

$$\mathrm{d}X_2(t) = \alpha X_2(t)\mathrm{d}t + \beta X_2(t)\mathrm{d}W_2(t) - \left(Au_1(t) + Au_2(t)\right) I_{\{X_2(t)>0\}}\mathrm{d}t. \tag{14}$$

The agents minimize

$$E\left(\left(X_1(1) + X_2(1)\right)^2 + \int_0^1 \left(u_1(s) + u_2(s)\right)^2 \mathrm{d}s \right). \tag{15}$$

$X_i(0) = x_i > 0$ for $i = 1, 2$ and $\alpha, A \geq 0$, $\beta \neq 0$. Now W_1, W_2 are independent or positively correlated Brownian motions, i.e. $\mathrm{d}W_1(t)\,\mathrm{d}W_2(t) = \rho \mathrm{d}t$ with $\rho \in [0;1]$. Moreover, we assume that the controls u_i, $i = 1, 2$, are so that $u_1(t) + u_2(t) \geq 0$ for $t \in [0;1]$ (we can not make negative improvements) and $E\int_0^1 \left(u_1(s) + u_2(s)\right)^{16} \mathrm{d}s$ is finite.

Let us comment on the dynamics given by (13)-(14). We assume that the pollution levels at the moment 0 are nonnegative numbers and when $A = 0$ they become geometric Brownian motions. When $A > 0$, the pollution levels X_1 and X_2 are reduced, as long as they are positive, by the improvements u_1 and u_2. When the level X_i, $i = 1, 2$, becomes 0 it stays equal 0 all the time and there is no need of reducing it. Since the reduction of the pollution is all the time proportional to the sum $u_1 + u_2$ we have **free transfer of technologies**, and the cost of improvements corresponds rather to the **cost of the development of technologies**, not the cost of implementing them on a specific domain - this could be proportional to the area of the domain. As a result, the agents act jointly as a one.

To minimize (15) we use similar techniques as before. Setting $u = u_1 + u_2$,

$$Z_F(t) = f(t)X_1^2(t) + f(t)X_2^2(t) + 2g(t)X_1(t)X_2(t) + \int_0^t u^2(s)\mathrm{d}s \tag{16}$$

and assuming that

$$\mathrm{d}f(t) = f'(t)\mathrm{d}t, \ \mathrm{d}g(t) = g'(t)\mathrm{d}t$$

(i.e. f and g are deterministic, absolutely continuous functions), we get that the local martingale part of Z_F is equal

$$2\beta f(t)X_1^2(t)\mathrm{d}W_1(t) + 2\beta f(t)X_2^2(t)\mathrm{d}W_2(t) + 2\beta g(t)X_1(t)X_2(t)\left\{\mathrm{d}W_1(t) + \mathrm{d}W_2(t)\right\}.$$

Since the solutions of (13) and (14) are given by $X_i(t) = U_i(t)V_i(t)I_{\{t<\tau_i\}}$, $i = 1, 2$, where

$$U_i(t) = \mathrm{e}^{\left(\alpha - \frac{1}{2}\beta^2\right)t + \beta W_i(t)}, \quad V_i(t) = x_i - A\int_0^t U_i^{-1}(s)\,u(s)\,\mathrm{d}s$$

and τ_i is the first time when X_i reaches zero, we may proceed as before and prove that under our choice of controls the process Z_F has a true martingale part.

The drift part of Z_F is equal

$$\begin{aligned}\left(\left(2\alpha+\beta^2\right)f+f'\right)\left(X_1^2+X_2^2\right)+2\left(\left(2\alpha+\rho\beta^2\right)g+g'\right)X_1X_2 \\ +u^2-2A\left(f+g\right)\left(X_1+X_2\right)u \\ \geq\left(\left(2\alpha+\beta^2\right)f+f'-A^2\left(f+g\right)^2\right)\left(X_1^2+X_2^2\right) \\ +2\left(\left(2\alpha+\rho\beta^2\right)g+g'-A^2\left(f+g\right)^2\right)X_1X_2.\end{aligned} \tag{17}$$

The drift is non-negative and Z_F is a submartingale when f and g satisfy the system of equations

$$\begin{cases} f'=A^2\left(f+g\right)^2-\left(2\alpha+\beta^2\right)f, \\ g'=A^2\left(f+g\right)^2-\left(2\alpha+\rho\beta^2\right)g. \end{cases} \tag{18}$$

The drift vanishes (Z_F is a martingale) for the specific choice of u

$$u=u^*=A\left(f+g\right)\left(X_1+X_2\right). \tag{19}$$

To notice that for the control given by formula (19) $E\int_0^1\left(u^*(s)\right)^{16}\mathrm{d}s<+\infty$, we may apply [1, Theorem 7.1.2].

Remark 2.2. We can not apply [1, Theorem 7.1.2] directly, since the system (13-14) with $u=u_1+u_2$ given by (19) has not globally Lipschitz coefficients in the space variable, but one may notice that for its solution X_i, $i=1,2$, and $t\in[0;1]$, $\int_0^1 X_i^{16}(s)\mathrm{d}s$ is less than $\int_0^1 X_{i,1}^{16}(s)\mathrm{d}s+\int_0^1 X_{i,2}^{16}(s)\mathrm{d}s$, where $X_{i,1}$, $i=1,2$, is the solution of

$$\begin{aligned}\mathrm{d}X_{1,1}(t)&=\alpha X_{1,1}(t)\mathrm{d}t+\beta X_{1,1}(t)\mathrm{d}W_1(t)-A^2\left(f(t)+g(t)\right)\left(X_{1,1}(t)+X_{2,1}(t)\right)\mathrm{d}t,\\ \mathrm{d}X_{2,1}(t)&=\alpha X_{2,1}(t)\mathrm{d}t+\beta X_{2,1}(t)\mathrm{d}W_2(t)-A^2\left(f(t)+g(t)\right)\left(X_{1,1}(t)+X_{2,1}(t)\right)\mathrm{d}t\end{aligned}$$

with starting values $X_{1,1}(0)=x_1$, $X_{2,1}(0)=x_2$, and $X_{i,2}$, $i=1,2$, is the solution of

$$\begin{aligned}\mathrm{d}X_{1,2}(t)&=\alpha X_{1,2}(t)\mathrm{d}t+\beta X_{1,2}(t)\mathrm{d}W_1(t+\tau)-A^2\left(f(t)+g(t)\right)X_{1,2}(t)\mathrm{d}t,\\ \mathrm{d}X_{2,2}(t)&=\alpha X_{2,2}(t)\mathrm{d}t+\beta X_{2,2}(t)\mathrm{d}W_2(t+\tau)-A^2\left(f(t)+g(t)\right)X_{2,2}(t)\mathrm{d}t\end{aligned}$$

with starting values $X_{1,2}(0)=X_{1,1}(\tau)$, $X_{2,2}(0)=X_{2,1}(\tau)$, where $\tau=\tau_1\wedge\tau_2\wedge 1$. Theorem 7.1.2 from [1] directly applies to $EX_{i,1}^{16}(s)$, while to apply it to $EX_{i,2}^{16}(s)$ one needs to justify that $EX_{i,1}^{16}(\tau)<+\infty$, which follows from the same reasoning as in the proof of [1, Theorem 7.1.2], where, instead of deterministic time t one considers (bounded) stopping time τ. Finally we get

$$E\int_0^1 X_{i,1}^{16}(s)\mathrm{d}s+E\int_0^1 X_{i,2}^{16}(s)\mathrm{d}s=\int_0^1 EX_{i,1}^{16}(s)\mathrm{d}s+\int_0^1 EX_{i,2}^{16}(s)\mathrm{d}s<+\infty.$$

Now, the fusion optimization problem will be solved if we show that the system (18), with the boundary conditions $f(1)=g(1)=1$, has a solution such that

$$A\left(f(t)+g(t)\right)\left(X_1(t)+X_2(t)\right)\geq 0 \text{ for } t\in[0,1] \tag{20}$$

(recall that we can not make negative improvements). Indeed, we can prove that the system (18) has a solution in f,g such that $f(1)=g(1)=1$ and

$$m\left(t\right):=f(t)+g(t)>0 \text{ for } t\in(-\infty,1], \tag{21}$$

see Lemma 2.7 in the Appendix, and since $A\geq 0$ and $X_1,X_2\geq 0$ we get (20).

REMARK 2.3. The solution of the optimisation problem which we have just obtained may be applied to a slightly more general range of parameters A, α, β and ρ. Namely (see Remark (2.8)), the solution of system (18) with the boundary conditions $f(1) = g(1) = 1$ exists and (20) holds for any $A \geq 0$, $\alpha, \beta \in \mathbb{R}$ and $\rho \in [-1; 1]$ such that $2\alpha + \rho\beta^2 \geq 0$. In the case $\beta = 0$ we obtain a deterministic optimal control problem.

2.3. Collusive actions. Now the dynamics of X_i, $i = 1, 2$, starting from $X_i(0) = x_i > 0$, is given by

$$\mathrm{d}X_i(t) = \alpha X_i(t)\mathrm{d}t + \beta X_i(t)\mathrm{d}W_i(t) - Au_i(t)I_{\{X_i(t)>0\}}\mathrm{d}t. \tag{22}$$

The assumptions on parameters α, A, β, and Brownian motions W_i, $i = 1, 2$, remain the same as in the fusion case. Controls u_i, $i = 1, 2$, are such that $E\int_0^1 u_i^{16}(s)\mathrm{d}s$, $i = 1, 2$, are finite.

From the form of (22) we see that when the level X_i, $i = 1, 2$, becomes 0 it stays equal 0 all the time and there is no need of reducing it. Thus $X_i \geq 0$, $i = 1, 2$. Moreover, we see that the free transfer of technologies does not occur in this case.

In the collusive actions framework the agents minimize

$$E\left((X_1(1) + X_2(1))^2 + \int_0^1 u_1^2(s)\mathrm{d}s + \int_0^1 u_2^2(s)\mathrm{d}s\right). \tag{23}$$

We proceed as in previous subsection but now we work with

$$Z_C = \varphi X_1^2 + \varphi X_2^2 + 2\psi X_1 X_2 + \int_0^{\cdot} u_1^2(s)\mathrm{d}s + \int_0^{\cdot} u_2^2(s)\mathrm{d}s. \tag{24}$$

Assuming that

$$\mathrm{d}\varphi(t) = \varphi'(t)\mathrm{d}t,\ \mathrm{d}\psi(t) = \psi'(t)\mathrm{d}t,$$

we get that the drift part of Z_C is equal to

$$\begin{aligned}
&\left(\left(2\alpha + \beta^2\right)\varphi + \varphi'\right)\left(X_1^2 + X_2^2\right) + 2\left(\left(2\alpha + \rho\beta^2\right)\psi + \psi'\right)X_1X_2 \\
&\quad + u_1^2 - 2A\left(\varphi X_1 + \psi X_2\right)u_1 + u_2^2 - 2A\left(\varphi X_2 + \psi X_1\right)u_2 \\
&\quad \geq \left(\left(2\alpha + \beta^2\right)\varphi + \varphi' - A^2\left(\varphi^2 + \psi^2\right)\right)\left(X_1^2 + X_2^2\right) \\
&\quad + 2\left(\left(2\alpha + \rho\beta^2\right)\psi + \psi' - 2A^2\varphi\psi\right)X_1X_2.
\end{aligned} \tag{25}$$

The drift is non-negative and Z_C is a submartingale (due to the choice of controls the local martingale part of Z_C is a true martingale) if φ and ψ satisfy the system of two equations

$$\begin{cases} \varphi' = A^2\left(\varphi^2 + \psi^2\right) - \left(2\alpha + \beta^2\right)\varphi, \\ \psi' = 2A^2\varphi\psi - \left(2\alpha + \rho\beta^2\right)\psi. \end{cases} \tag{26}$$

The drift vanishes (Z_C is a martingale) if additionally $u_1 = u_1^* = A\left(\varphi X_1 + \psi X_2\right)$ and $u_2 = u_2^* = A\left(\varphi X_2 + \psi X_1\right)$. These controls are admissible.

We set the boundary conditions $\varphi(1) = \psi(1) = 1$. Now, the problem of minimization of (23) will be solved if we show that the system (26) (with the boundary conditions $\varphi(1) = \psi(1) = 1$) has a solution such that $\varphi \geq 0$ and $\psi \geq 0$ for $t \in [0; 1]$, and then

$$u_1^* = A\left(\varphi X_1 + \psi X_2\right) \geq 0,$$
$$u_2^* = A\left(\varphi X_2 + \psi X_1\right) \geq 0.$$

But by Lemma 2.9 (see the Appendix) the solution of system (26) with the boundary conditions $\varphi(1) = \psi(1) = 1$ exists for $t \in (-\infty; 1]$, and for such t we have $\varphi(t) > 0$ and $\psi(t) > 0$.

Remark 2.4. Since (see Remark 2.10), the solution of system (26) with the boundary conditions $\varphi(1) = \psi(1) = 1$ and with $\varphi(t) > 0$, $\psi(t) > 0$ for $t \in (-\infty; 1]$ exists for any $A, \alpha, \beta \in \mathbb{R}$ and $\rho \in [-1; 1]$, the solution of the optimisation problem which we have just obtained may be applied for any parameters $A \geq 0$, $\alpha, \beta \in \mathbb{R}$ and $\rho \in [-1; 1]$.

Remark 2.5. We have tried to solve these problems using the technique of Cadenillas et al. [2], for $\beta = \alpha$, and

$$Z_i(t) = \begin{cases} e^{-W_i(t) - \frac{1}{2}t}, & \text{if } X_i(t) > 0, \\ 0, & \text{if } X_i(t) = 0. \end{cases}$$

For example, in the collusive case we proceeded to minimize separately:

$$E\left(\int_0^1 \left(u^2(s) - \lambda u(s) Z_i(s)\right) \mathrm{d}s\right)$$

and

$$E\left(X_i^2(1) + E^2 X_i(1) - \lambda Z_i(1) X_i(1)\right)$$

with Lagrange multiplier λ, setting $X_i(1) = \lambda Y_i(1)$, $Y_i = (Z_i - k)_+$ where $k = E(Z_i - k)_+$. But following the remark of Freddy Delbaen (private communication) this minimum clearly can not be attained in this setting . Using this technique we can only show that for small β and A the fusion solution generates better results, just comparing this "solution" with the deterministic one ($\beta = 0$) and using continuity.

2.4. Conclusion and a numerical example. In this subsection we summarize the obtained results and provide numerical example. As a result of calculations done in the preceding subsections we have

Theorem 2.6. *Let $X_i(0) = x_i > 0$, $i = 1, 2$, be the pollution levels in the two domains at the moment $t = 0$. Moreover, assume that the dynamics of X_i for $t \in [0; 1]$ is given by one of the three possible scenarios: separate actions (1), fusion (13)-(14) or collusion (22), where $\alpha, A \geq 0$, $\beta \neq 0$, $u_1, u_2 \geq 0$ and W_1, W_2 are independent or positively correlated Brownian motions, i.e. $\mathrm{d}W_1(t)\,\mathrm{d}W_2(t) = \rho \mathrm{d}t$ with $\rho \in [0; 1]$. The minimal values of (2), (15) and (23), obtained by the optimal control, are given respectively by*

- *in the separate actions case*
$$y(0)\, x_i^2;$$
- *in the fusion case*
$$f(0)\left(x_1^2 + x_2^2\right) + 2g(0)\, x_1 x_2;$$
- *in the collusive actions case*
$$\varphi(0)\left(x_1^2 + x_2^2\right) + 2\psi(0)\, x_1 x_2,$$

where y, (f, g) and (φ, ψ) satisfy (7), (18) and (26) with boundary conditions $y(1) = 1$, $(f(1), g(1)) = (1, 1)$ and $(\varphi(1), \psi(1)) = (1, 1)$ respectively.

It appears, that in some range of the parameters α, A, β and ρ the fusion case generates lower costs than the collusive actions case. We have for example

	\Model parameters Quantity\	$\alpha = 1, \beta = 0.7$ $A = 0.7, \rho = 0.7$	$\alpha = 1, \beta = 2$ $A = 2, \rho = 0$
Separate actions	$y(0)$	3.80	1.5
Fussion	$f(0), g(0)$	1.55, 0.87	16.7, −13.81
Collusive optima	$\varphi(0), \psi(0)$	2.59, 1.78	1.49, 0.00

Appendix.

LEMMA 2.7. *When $A, \alpha \geq 0$, $\rho \in [0;1]$ and $\beta \neq 0$ then the solution of system (18) with boundary conditions $f(1) = g(1) = 1$ exists for $t \in (-\infty; 1]$ and for such t we have*

$$f(t) + g(t) > 0.$$

Proof. First, consider the system (18) when $\rho = 1$ (i.e. $W_1 = W_2$). In this case we get the solution given by the formula

$$f(t) = g(t) = \frac{1}{4\frac{A^2}{B}\left(1 - e^{B(t-1)}\right) + e^{B(t-1)}} > 0, \tag{27}$$

where $B = 2\alpha + \beta^2 > 0$.

Now we will assume that $\rho < 1$. Let us notice that due to Peano's theorem (cf. [3, Theorem 2.19]) we know that the solution exists on some interval $(a;1]$ with $a < 1$. Let us consider the sum $m(t) := f(t) + g(t)$. Assume that for some $t_1 \in (a;1]$, $m(t_1) \leq 0$. By the continuity argument and $m(1) = 2 > 0$, we get that for some $t_0 \in [t_1;1)$,

$$m(t_0) = 0, \tag{28}$$

and for $t \in (t_0;1)$, $m(t) > 0$. By (18) we see that

$$m' = 2A^2m^2 - \left(2\alpha + \rho\beta^2\right)m - (1-\rho)\beta^2 f.$$

and for $t = t_0$ we have

$$m'(t_0) = -(1-\rho)\beta^2 f(t_0),$$

hence, in order to cross the level 0 we must have $f(t_0) \leq 0$. Hence

$$f(t_0) = -g(t_0) \leq 0. \tag{29}$$

Now let us consider the difference $h = f - g$. We have

$$h'(t) = -\left(2\alpha + \rho\beta^2\right)h(t) - (1-\rho)\beta^2 f(t).$$

Let $t_2 \in [t_0;1)$ be such that $h(t_2) = 0$ and for $t \in (t_2;1)$, $h(t) \geq 0$. Such t_2 exists, since $h(t_0) = 2f(t_0) \leq 0$, $h(1) = 0$ and $h'(1) = -(1-\rho)\beta^2 < 0$, so h is decreasing in the neighborhood of 1. Hence there exists $t_3 \in (t_2;1)$ (local maximum) such that

$$h'(t_3) = -\left(2\alpha + \rho\beta^2\right)h(t_3) - (1-\rho)\beta^2 f(t_3) = 0. \tag{30}$$

Now, from $h(t_3) \geq 0$, $2\alpha + \rho\beta^2 \geq 0$ and (30) we get that

$$f(t_3) \leq 0. \tag{31}$$

But from $h(t_3) \geq 0$ and $f(t_3) = h(t_3) + g(t_3)$ we get

$$f(t_3) \geq g(t_3). \tag{32}$$

Now (31) and (32) contradict that for $t \in (t_0; 1)$, $m(t) = f(t) + g(t) > 0$. Thus, the assumption that for some $t_1 \in (a; 1]$, $u(t_1) \leq 0$ leads to contradiction and

$$f(t) + g(t) > 0 \text{ for all } t \in (a; 1].$$

Now we will prove that the solution exists on the whole interval $(-\infty; 1]$. First, notice that we must have

$$f(t) > 0 \text{ for all } t \in (a; 1]. \tag{33}$$

Indeed, if $f(t_4) \leq 0$ for some $t_4 \in (a; 1]$, by the just proved fact that for all $t \in (a; 1]$, $f(t) + g(t) > 0$, we would have

$$f(t_4) \leq 0, \;\; g(t_4) > 0. \tag{34}$$

But now, reasoning in a similar way as before (from (29) onwards, but using (34) instead of (29) to see that $h(t_4) \leq 0$) we again get a contradiction. Notice now that system (18) is equivalent to the system

$$\begin{cases} f' = A^2 (2f - h)^2 - \left(2\alpha + \beta^2\right) f, \\ h' = -\left(2\alpha + \rho\beta^2\right) h - (1 - \rho) \beta^2 f. \end{cases} \tag{35}$$

We have that

$$\lim_{t \downarrow a} f(t) < +\infty. \tag{36}$$

This follows from the first equation, since $f' \geq -\left(2\alpha + \beta^2\right) f$, thus f has no faster than exponential decline. Since we know that $f(t) > 0$ for $t \in (a; 1]$, $\lim_{t \downarrow a} f(t) \geq 0$. But by this, (36) and $h' = -\left(2\alpha + \rho\beta^2\right) h - \beta^2 f$ we see that

$$\lim_{t \downarrow a} h(t) \in (-\infty; +\infty).$$

Hence, using Peano's theorem we may extend the interval when the solutions of (35) (and hence (18)) exist. Thus the solution of (18) exists on the interval $(-\infty; 1]$. ■

Remark 2.8. The restrictions imposed on the parameters A, α, β and ρ may be slightly weakened. When $\beta = 0$ and $B = 2\alpha + \beta^2 \neq 0$ we get the solution given by (27); when $(1 - \rho) \beta = 0$ and $B = 2\alpha + \beta^2 = 0$ we get the solution $f(t) = g(t) = \left(1 + 4A^2 (1 - t)\right)^{-1}$; and for $\rho < 1$ in the proof of Lemma 1 we were using only the relations $2\alpha + \rho\beta^2 \geq 0$, $\beta \neq 0$ and $\rho \in [-1; 1]$. Thus, Lemma 1 remains true for any $A, \alpha, \beta \in \mathbb{R}$ and $\rho \in [-1; 1]$ such that $2\alpha + \rho\beta^2 \geq 0$.

Now we will prove

Lemma 2.9. *With the assumptions on parameters A, α, β and ρ as in Lemma 2.7, the solution of system (26) with starting conditions $\phi(1) = \psi(1) = 1$ exists for $t \in (-\infty; 1]$ and for such t we have*

$$\phi(t) > 0, \;\; \psi(t) > 0.$$

Proof. Again, by Peano's theorem we know that the solution exists on some interval $(a; 1]$ with $a < 1$. By $\psi' = 2A^2 \phi\psi - \left(2\alpha + \rho\beta^2\right) \psi$ and $\psi(1) = 1$ for $t \in (a; 1]$ we get

$$\psi(t) = \exp\left(-2A^2 \int_t^1 \phi(s) \, \mathrm{d}s + \left(2\alpha + \rho\beta^2\right)(1 - t)\right) > 0.$$

Consider the difference $\chi = \phi - \psi$. We have

$$\chi' = A^2\chi^2 - \left(2\alpha + \beta^2\right)\chi - (1-\rho)\,\beta^2\psi.$$

Notice that for $\rho = 1$ we have the following solution $\chi \equiv 0$ for $t \in (a;1]$ and thus $\varphi = \psi > 0$ for $t \in (a;1]$.

Now let $\rho < 1$ and let us assume that for some $t_0 \in (a;1]$,

$$\phi\,(t_0) < 0.$$

Let $t_2 \in (t_0;1)$ be such that $\chi\,(t_2) = 0$ and for $t \in (t_2;1)$,

$$\chi\,(t) \geq 0. \tag{37}$$

Such t_2 exists, since $\chi\,(t_0) = \phi\,(t_0) - \psi\,(t_0) < 0$, $\chi\,(1) = 0$ and $\chi'\,(1) = -\,(1-\rho)\,\beta^2 < 0$, so χ is decreasing in the neighborhood of 1. But we see that

$$\chi'\,(t_2) = -\,(1-\rho)\,\beta^2\psi\,(t_2) < 0. \tag{38}$$

Hence there exists $\varepsilon > 0$ such that for $t \in (t_2; t_2+\varepsilon)$

$$\chi\,(t) < 0. \tag{39}$$

But this contradicts (37).

The existence of the solution on whole interval $(-\infty;1]$ follows from similar reasoning as in the proof of Lemma 1. ∎

REMARK 2.10. In fact, Lemma 2 holds for any real parameters A, α, β and $\rho \in [-1;1]$. When $(1-\rho)\,\beta = 0$ we get the solution $\varphi\,(t) = \psi\,(t) = \left(1 + 2A^2\,(1-t)\right)^{-1}$ for $B = 2\alpha + \beta^2 = 0$ and

$$\varphi\,(t) = \psi\,(t) = \frac{1}{2\frac{A^2}{B}\left(1 - \mathrm{e}^{B(t-1)}\right) + \mathrm{e}^{B(t-1)}}$$

for $B = 2\alpha + \beta^2 \neq 0$. Moreover, for $\rho < 1$ in the proof of Lemma 1 we were using only the relations $\beta \neq 0$ and $\rho \in [-1;1]$. Thus, Lemma 2 remains true for any $A, \alpha, \beta \in \mathbb{R}$ and $\rho \in [-1;1]$.

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