Money in the utility model

Monetary Economics

Michał Brzoza-Brzezina

Warsaw School of Economics
Plan of the Presentation

1. Motivation
2. Model
3. Steady state
4. Short run dynamics
5. Simulations
6. Extensions
7. Conclusions
Introduction

- The neoclassical growth model by Ramsey (1928) and Solow (1956) provides the basic framework for modern macroeconomics.

- But these are models of nonmonetary economies.

- In order to explain monetary policy we have to introduce a monetary policy instrument.

- Sidrauski (1967) introduced money into the neoclassical growth model.

- At this time money was considered the monetary policy instrument.

- The model derives from Ramsey (1928): utility maximization by the representative agent.
Introduction (cont’d)

- How can we introduce money?
  - find explicit role for money (e.g. money is necessary to make transactions - Cash in Advancec (CIA) approach, Clower (1967))
  - assume money yields utility (Money in the Utility Function (MIU) - Sidrauski (1967))

- MIU is simple though not very appealing. Can be rationalized by transaction demand for money.

- The model allows us studying:
  - impact of money (i.e. monetary policy) on the real economy,
  - impact of money on prices,
  - optimal rate of inflation.

- Derivation follows (approximately) Walsh (2003)
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1. Motivation
2. Model
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6. Extensions
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Household’s problem - objective

Suppose the utility function of the representative household is:

\[ U_t = u(c_t, m_t) \]  

(1)

where \( m_t = \frac{M_t}{P_t} \)

and utility is increasing and strictly concave in both arguments.

The representative household seeks to maximize lifetime utility:

\[ U = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, m_t) \]  

(2)
subject to the budget constraint:

\[ y_t + \tau_t + (1 - \delta)k_{t-1} + \frac{i_{t-1}B_{t-1}}{P_t} + \frac{M_{t-1}}{P_t} = (3) \]

\[ c_t + k_t + \frac{M_t}{P_t} + \frac{B_t}{P_t} \]

and the production function:

\[ y_t = f(k_{t-1}) \]

This means that households’ income can be spent on consumption, invested as capital, saved as bonds or kept as money.
We can substitute the production function and rewrite the budget constraint:

\[
    f(k_{t-1}) + \tau_t + (1 - \delta)k_{t-1} + \frac{i_{t-1}b_{t-1} + m_{t-1}}{\pi_t} = c_t + k_t + m_t + b_t
\]

where \( \pi_t \equiv \frac{P_t}{P_{t-1}} \).
Lagrangian

\[
\max_{c,m,k,b} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t, m_t) + \lambda_t \left[ f(k_{t-1}) + \tau_t + (1 - \delta) k_{t-1} + \frac{i_{t-1} b_{t-1} + m_{t-1}}{\pi_t} \right] - c_t - k_t - m_t - b_t \right\} 
\]

(5)
First order conditions for this problem are:

$c_t$:

$$u'(c_t) = \lambda_t$$  \hspace{1cm} (6)

$k_t$:

$$-\lambda_t + E_t \lambda_{t+1} \beta [f'(k_t) + (1 - \delta)] = 0$$  \hspace{1cm} (7)

$m_t$:

$$u'(m_t) - \lambda_t + E_t \lambda_{t+1} \frac{\beta}{\pi_{t+1}} = 0$$  \hspace{1cm} (8)

$b_t$:

$$-\lambda_t + E_t \lambda_{t+1} \beta \frac{i_t}{\pi_{t+1}} = 0$$  \hspace{1cm} (9)
FOCs can be transformed into formulae describing allocation choices in equilibrium. Substitute (6) into (9):

$$u'(c_t) = \beta E_t u'(c_{t+1}) \frac{i_t}{\pi_{t+1}}$$

(10)

This is the Euler equation - key intertemporal condition in general equilibrium models. It determines allocation over time. Utility lost from consumption today equals utility from consuming tomorrow adjusted for the (real) gain from keeping bonds.
Equilibrium conditions - equality of expected returns

Substitute (9) into (7):

\[ E_t u'(c_{t+1})[f'(k_t) + (1 - \delta)] = E_t u'(c_{t+1}) \frac{i_t}{\pi_{t+1}} \] \hspace{1cm} (11)

Define real interest rate (Fisher equation):

\[ r_t \equiv E_t \frac{i_t}{\pi_{t+1}} \] \hspace{1cm} (12)

The marginal product of capital (net of depreciation) equals the real interest rate (up to first order).
Equilibrium conditions - Opportunity cost of money

Merge (6), (9) and (8):

\[ u'(m_t) - u'(c_t) + \frac{u'(c_t)}{i_t} = 0 \]

Rearrange:

\[ \frac{u'(m_t)}{u'(c_t)} = 1 - \frac{1}{i_t} \]  \hspace{1cm} (13)

This equates the marginal rate of substitution between money and consumption to their relative price.
Technology and utility

In order to solve the model we have to assume a production and utility function.

Production function:

\[ y_t = e^{z_t} k_{t-1} \]  \hspace{1cm} (14)

where

\[ z_t = \rho_z z_{t-1} + \varepsilon_{z,t} \]  \hspace{1cm} (15)

is a stochastic TFP process.

Utility function:

\[ U_t = \ln c_t + \ln m_t \]  \hspace{1cm} (16)
Monetary policy is very simple. We assume that money follows a stochastic process:

\[ M_t = e^{\theta_t} M_{t-1} \]

which yields:

\[ m_t = \frac{m_{t-1}}{\pi_t} e^{\theta_t} \tag{17} \]

where:

\[ \theta_t = \rho_\theta \theta_{t-1} + \epsilon_{\theta,t} \tag{18} \]

Is a stochastic money supply process.
Government and bonds

Bonds are in zero net supply

\[ b = 0 \]

and the Government budget is balanced every period

\[ \tau_t P_t = M_t - M_{t-1} \]

in real terms

\[ \tau_t = m_t - \frac{m_{t-1}}{\pi_t} \]

Combine these with the household budget constraint (4):

\[ y_t + (1 - \delta) k_{t-1} = c_t + k_t \]
The model

We now have 9 equations:
\((10), (11), (12), (13), (14), (15), (17), (18), (19)\)
to solve for 9 endogeneous variables:
\(y_t, k_t, m_t, c_t, i_t, r_t, \pi_t, z_t, \theta_t\).
This will be done in two steps:

1. Find the model’s steady state,
2. Derive log-linear approximation around it.
## Plan of the Presentation

1. Motivation
2. Model
3. Steady state
4. Short run dynamics
5. Simulations
6. Extensions
7. Conclusions
The steady state is where the model economy converges to (stabilizes) in the absence of shocks. In our model there is no population or productivity growth, so in the steady state output, consumption etc. will be constant. How do we solve for the steady state?

1. assume shocks are zero,
2. drop time indices \((x_{t-1} = x_t = x_{ss})\).
The steady state - capital

The steady state is where the model economy converges to (stabilizes) in the absence of shocks. In our model there is no population or productivity growth, so in the steady state output, consumption etc. will be constant. From (11), (10) and (14) we have:

\[
\alpha (k^{ss})^{\alpha - 1} = \frac{1}{\beta} - 1 + \delta
\]

or

\[
k^{ss} = \left[ \frac{1}{\alpha} \left( \frac{1}{\beta} - 1 + \delta \right) \right]^{\frac{1}{\alpha - 1}} \tag{20}
\]

This determines capital (and output) in the SS.
The steady state - consumption

Take (19) and (14). This yields:

\[ c^{ss} = (k^{ss})^\alpha - \delta k^{ss} \quad (21) \]

Since \( k^{ss} \) is given by (20), this determines steady state consumption. Note that (abstracting away from population and technology growth) this equals the steady state of the Ramsey economy.
The steady state - money

From (13):

$$m^{ss} = c^{ss} \frac{i^{ss}}{i^{ss} - 1}$$

From Euler (10)

$$i^{ss} = \pi^{ss} / \beta$$

so that:

$$m^{ss} = c^{ss} \frac{\pi^{ss}}{\pi^{ss} - \beta}$$  \hspace{1cm} (22)
Neutralit y of money

- Definition of (long run) neutrality: in the long run real variables do not depend on nominal variables.
- This is true for all variables but real money holdings.
- But the latter are usually excluded from the definition.
- So, in the MIU money is neutral in the long run.
Plan of the Presentation

1. Motivation
2. Model
3. Steady state
4. **Short run dynamics**
5. Simulations
6. Extensions
7. Conclusions
Short run dynamics

- Steady state tells us about the long run
- But most interesting in most applications are short run dynamics
- We have a set of difference equations that describe it
- But they are non-linear: difficult to solve, to estimate etc.
- We will assume that the economy is always in the vicinity of the steady state
- This seems a reasonable assumption when analysing short-run (business cycle) dynamics
- Then we take a log-linear approximation around the steady state which is relatively simple to handle
- The procedure of log-linearization is explained in Walsh (p. 85-87)
Log-linearization

Log-linear approximation. Define $\hat{x}_t \equiv \ln \left( \frac{x_t}{x^{ss}} \right)$ as log-deviation from steady state.

It follows that (first order Taylor expansion around $\hat{x}_t = 0$):

$$x_t = x^{ss} \frac{x_t}{x^{ss}} = x^{ss} e^{\hat{x}_t} \simeq x^{ss} e^0 + x^{ss} e^0 (\hat{x}_t - 0) = x^{ss} (1 + \hat{x}_t)$$

This can be used to derive the approximation. Some useful tricks:

$$x_t y_t = x^{ss} y^{ss} (1 + \hat{x}_t + \hat{y}_t) \quad (23)$$

$$\frac{x_t}{y_t} = \frac{x^{ss}}{y^{ss}} (1 + \hat{x}_t - \hat{y}_t) \quad (24)$$

$$x_t^a = (x^{ss})^a (1 + a\hat{x}_t) \quad (25)$$

$$\ln x_t = \ln x^{ss} + \hat{x}_t \quad (26)$$
\[ c_t^{-1} = \beta E_t \frac{r_t}{c_{t+1}} \]

\[ (c^{ss})^{-1} (1 - \hat{c}_t) = \beta \frac{r^{ss}}{c^{ss}} E_t (1 + \hat{r}_t - \hat{c}_{t+1}) \]

**Steady state;**

\[ (c^{ss})^{-1} = \beta \frac{r^{ss}}{c^{ss}} \] (27)

**Divide:**

\[ \hat{c}_t = E_t (\hat{c}_{t+1} - \hat{r}_t) \] (28)

LL Euler
LL budget constraint

\[ y_t + (1 - \delta) k_{t-1} = c_t + k_t \]

\[ y^{ss}(1 + \hat{y}_t) + (1 - \delta) k^{ss}(1 + \hat{k}_{t-1}) = c^{ss}(1 + \hat{c}_t) + k^{ss}(1 + \hat{k}_t) \]

Steady state:

\[ y^{ss} + (1 - \delta) k^{ss} = c^{ss} + k^{ss} \]

Substract:

\[ y^{ss} \hat{y}_t + (1 - \delta) k^{ss} \hat{k}_{t-1} = c^{ss} \hat{c}_t + k^{ss} \hat{k}_t \]

Rearrange:

\[ \hat{k}_t = (1 - \delta) \hat{k}_{t-1} + \frac{y^{ss}}{k^{ss}} \hat{y}_t - \frac{c^{ss}}{k^{ss}} \hat{c}_t \]
LL production function

\[ y_t = e^{z_t} k_t^\alpha \]

\[
\begin{align*}
\ln y_t &= z_t + \alpha \ln k_{t-1} \\
\ln y^{ss} + \hat{y}_t &= z_t + \alpha \ln k^{ss} + \alpha \hat{k}_{t-1} \\
y^{ss} &= (k^{ss})^\alpha \\
\hat{y}_t &= \alpha \hat{k}_{t-1} + z_t
\end{align*}
\] (29)
\[ r_t \equiv E_t \frac{i_t}{\pi_{t+1}} \]

\[ \hat{r}_t \equiv \hat{i}_t - E_t \hat{\pi}_{t+1} \] (30)
LL money rule (money supply)

\[ m_t = \frac{m_{t-1}}{\pi_t} e^{\theta_t} \]

\[ \ln m^{ss} + \hat{m}_t = \ln m^{ss} + \hat{m}_{t-1} - \ln \pi^{ss} - \hat{\pi}_t + \theta_t \]

\[ \hat{m}_t = \hat{m}_{t-1} - \hat{\pi}_t + \theta_t \] (31)
LL opportunity cost (money demand)

\[
\frac{c_t}{m_t} = 1 - \frac{1}{i_t}
\]

\[
\frac{c^{ss}}{m^{ss}} (1 + \hat{c}_t - \hat{m}_t) = 1 - \frac{1}{i^{ss}} (1 - \hat{i}_t)
\]

Steady state:

\[
\frac{c^{ss}}{m^{ss}} = \frac{i^{ss} - 1}{i^{ss}}
\]

\[
\hat{c}_t - \hat{m}_t = \frac{\hat{i}_t}{i^{ss} - 1}
\]
LL equality of returns

\[
E_t \left[ \frac{1}{c_{t+1}} \left[ \alpha e^{z_{t+1}} k_{t}^{\alpha-1} + (1 - \delta) \right] \right] = E_t \left[ \frac{1}{c_{t+1}} r_t \right]
\]

Denote \( x_t \equiv \alpha e^{z_t} k_{t-1}^{\alpha-1} + (1 - \delta) = \alpha \frac{y_t}{k_{t-1}} + 1 - \delta \)

\[
E_t \left[ \frac{1}{c_{t+1}} x_{t+1} \right] = E_t \left[ \frac{1}{c_{t+1}} r_t \right]
\]

\[
\frac{\chi^{ss}}{C^{ss}} E_t (1 - \hat{c}_{t+1} + \hat{x}_{t+1}) = \frac{r^{ss}}{C^{ss}} E_t [1 - \hat{c}_{t+1} + \hat{r}_t]
\]

Divide by steady state \( \frac{\chi^{ss}}{C^{ss}} = \frac{r^{ss}}{C^{ss}} \)

\[
E_t \hat{x}_{t+1} = E_t \hat{r}_t
\]
LL equality of returns cont’d

Now go back to $x_t \equiv \alpha \frac{y_t}{k_{t-1}} + 1 - \delta$

$$x^{ss} E_t (1 + \hat{x}_{t+1}) = \alpha \frac{y^{ss}}{k^{ss}} E_t (1 + \hat{y}_{t+1} - \hat{k}_{t}) + 1 - \delta$$

substitute for $E_t \hat{x}_{t+1}$ and $x^{ss}$

$$r^{ss} E_t (1 + \hat{r}_{t}) = \alpha \frac{y^{ss}}{k^{ss}} E_t (1 + \hat{y}_{t+1} - \hat{k}_{t}) + 1 - \delta$$

substract steady state

$$r^{ss} = \alpha \frac{y^{ss}}{k^{ss}} + 1 - \delta$$

$$r^{ss} E_t \hat{r}_{t} = \alpha \frac{y^{ss}}{k^{ss}} E_t (\hat{y}_{t+1} - \hat{k}_{t})$$
Again, we have 9 equations in 9 variables,

- We have some additional parameters \( r^{ss}, \pi^{ss}, i^{ss} \) that have to be calculated,

From (27): \( r^{ss} = \beta^{-1} \)
From (17): \( \pi^{ss} = 1 \)
From (12): \( i^{ss} = \pi^{ss} r^{ss} = \beta^{-1} \)

- Now we can solve the model e.g. in DYNARE and check its properties.
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1. Motivation
2. Model
3. Steady state
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5. Simulations
6. Extensions
7. Conclusions
Simulations in DYNARE

- Simulations will be done in DYNARE
- What is DYNARE?
  - Software (free :-) for solving, simulating and estimating general equilibrium models
  - www.dynare.org
- But requires a computing platform: MATLAB (expensive) or Octave (free)
We have to assume parameter values. These are taken from the literature.

\[ \beta = .99 \] (Note that in steady state this is the reciprocal of the (annualised) real interest rate of 4%)

\[ \alpha = .33 \] (Capital share)

\[ \delta = .025 \] (Depreciation approx. 10% on annual basis)

\[ \rho_z = \rho_\theta = .9 \] (Some inertia)
IRFs to a productivity shock
IRFs to a money supply shock

$m$

$i \times 10^{-3}$

$\pi$

$\theta$
Productivity shocks have an impact on the real economy and drive the business cycle in the MIU model.

Monetary shocks have an impact on nominal variables: inflation and nominal interest rate.

But monetary shocks affect only real money balances, but no other real variables.

In the MIU model money is neutral also in the short run.
MIU - exercises

- Impulse responses
  - generate impulse responses to monetary and productivity shocks
  - change selected parameters (e.g. $\beta$ or $\delta$). Does it affect our conclusions about neutrality of money?

- Calibrating business cycle properties of the MIU model
  - take the linearized version of the MIU model (MIU.mod)
  - calibrate the standard deviation and autoregression of the productivity shock $\varepsilon_{z,t}$ to match the respective moments for GDP in US data.
  - see what happens if productivity $z_t$ is assumed i.i.d. Compare impulse responses and moments with the calibration above.
Steady state calibration

MIU_level.mod implements the nonlinear version of the MIU model

Recalibrate ss real interest rate:
- check what is the steady state real interest rate
- assume that instead the real interest rate is 6%
- recalculate the model accordingly

Recalibrate ss investment rate
- check what is the steady state investment rate
- assume that instead the investment rate should be 30%
- recalculate the model accordingly
Plan of the Presentation

1. Motivation
2. Model
3. Steady state
4. Short run dynamics
5. Simulations
6. Extensions
7. Conclusions
Optimal rate of inflation

- One of the hot topics in monetary economics is the optimal rate of inflation.
- This is not only a theoretical but also a very practical question: Central Banks have to choose inflation targets.
- What does the MIU model tell us about the optimal rate of inflation?
- Utility depends on consumption and real money balances.
- The only impact inflation has on utility is via real money balances.
For easiness of exposition let’s restrict attention to steady state. Look at (22)

\[ m^{ss} = c^{ss} \frac{\pi^{ss}}{\pi^{ss} - \beta} = c^{ss}(1 + \frac{\beta}{\pi^{ss} - \beta}) \]

We now that \( c^{ss} \) is independent of inflation. So inflation reduces real money balances and, hence, utility.

Optimal is minimal rate of inflation. Given (12) and the requirement \( i \geq 0 \) we have:

\[ \pi^{OPT} = -r \]

So the optimal rate of inflation (in the MIU model) is deflation at rate \(-r\). This was postulated by M. Friedman.
Our model is neutral,

But it is possible to generate nonneutrality in the MIU model,

One possibility: add labour-leisure choice and nonseparable preferences,

Walsh (2003) shows the solution and simulations,

Fig 2.3 and 2.4 from textbook,

But this model cannot, under standard calibration, generate reaction functions and moments similar to those observed in the data.
Exercises - optimal rate of inflation

- Optimal rate of inflation
  - take the nonlinear version of the MIU model (`MIU_level.mod`).
  - add a variable that calculates welfare
  - check how steady state welfare depends on the inflation rate
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1. Motivation
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4. Short run dynamics
5. Simulations
6. Extensions
7. Conclusions
MIU - conclusions

- We solved and simulated a simple monetary dynamic stochastic general equilibrium model (DSGE),
- This was able to reflect some desired business cycle features in reaction to productivity shocks,
- But monetary shocks have only nominal effects (except for impact on real money balances),
- This is inconsistent with stylised facts presented in Lecture 1,
- We have to look for other models,
- E.g. the sticky price new Keynesian model - next lecture.