Solving DSGE models: an example.
Hansens Real Business Cycle Model
IAMA, Lecture 5

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Outline

1. The solution strategy
   - Overview

2. Hansen's benchmark Real Business Cycle Model
   - The model
   - Rational expectations
   - Labor supply

3. The solution steps
   - Step 1: find the FONCs
   - Step 2: Calculate the steady state
   - Step 3: Loglinearize
   - Step 4: Solve for the RLOM
   - Step 5: Calculate impulse responses

4. Representations
   - Alternative representations
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The solution strategy for a model works as follows:

1. Find the first order necessary conditions
2. Calculate the steady state
3. Loglinearize around the steady state
4. Solve for the recursive law of motion
5. Calculate impulse responses and (HP-filtered) moments

We will execute this strategy, using Hansens real business cycle model as particular example.
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Hansens benchmark Real Business Cycle Model

\[
\max E \left[ \sum_{t=0}^{\infty} \beta^t (\log c_t - An_t) \right]
\]

s.t.
\[
c_t + k_t = \bar{\gamma} e^{zt} k_{t-1}^\theta n_t^{1-\theta} + (1 - \delta) k_{t-1}
\]

and
\[
z_t = \rho z_{t-1} + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma^2) \text{ i.i.d.}
\]

where \( c_t \) is consumption, \( n_t \) is labor, \( k_t \) is capital, \( \gamma_t = \bar{\gamma} e^{zt} \) is total factor productivity (TFP).
Define, for convenience:

**output:** \[ y_t = \bar{\gamma} e^{z_t} k_{t-1}^\theta n_t^{1-\theta} \]

**return:** \[ R_t = \theta \frac{y_t}{k_{t-1}} + 1 - \delta \]

See:

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Rational expectations

- We assume that the social planner chooses $c_t$, $k_t$, $n_t$ etc., using all available information at date $t$, and forming **rational expectations** about the future.

- Rational expectations are the mathematical expectations, using all available information.

- Rational expectations only "live in" a model, in which the stochastic nature of all variables is clearly spelled out.
Rational expectations

Example: dice role.

- Dice 1, date $t$: $X_t$. Dice 2, date $t+1$: $Y_{t+1}$. Sum:
  \[ S_{t+1} = X_t + Y_{t+1}. \]
- $E_{t-1}[S_{t+1}] = 7$. $E_t[S_{t+1}] = 3.5 + X_t$.
- $E_{t+1}[S_{t+1}] = X_t + Y_{t+1}$.
- E.g. $X_t = 2$, $Y_{t+1} = 1$. Then $E_{t-1}[S_{t+1}] = 7$, $E_t[S_{t+1}] = 5.5$, $E_{t+1}[S_{t+1}] = 3$.

Example: AR(1)

- $z_{t+1} = \rho z_t + \epsilon_{t+1}$, $E_t[\epsilon_{t+1}] = 0$.
- Then: $E_t[z_{t+1}] = \rho z_t$. 

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Example: AR(1)
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Labor lotteries and labor supply

- We assume a very **elastic** labor supply for aggregate labor $n_t$,

$$u_t = \log(c_t) - An_t$$

- ... which turns out to be needed in order to quantitatively explain observed employment fluctuations.
- However, we typically imagine individual labor elasticity to be small.
- This can be true simultaneously by considering **labor lotteries**.
Labor lotteries and labor supply

- Individual labor supply $\tilde{n}_t$ may be based on some utility function $u(c_t) + v(\tilde{n}_t)$.
- Suppose that
  - labor is **indivisible**: agents either have a job or do not, $\tilde{n}_t = 0$ or $\tilde{n}_t = n^*$.
  - Agents are assigned to jobs according to a lottery, with probability $\pi_t$.
  - Shirking, moral hazard etc. are not possible. Unemployment insurance is perfect, and consumption $c_t$ is independent of job status.
- Total labor supplied: $n_t = \pi_t n^*$
- Normalization: $v(0) = 0$, $v(n^*)/n^* =: -A < 0$.
- Expected utility:
  \[
  E[u(c_t) + v(\tilde{n}_t)] = u(c_t) + \pi_t v(n^*) = u(c_t) - An_t
  \]
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Step 1: Find the first-order necessary conditions (FONCS)

Form the Lagrangian

\[
L = \max E \left[ \sum_{t=0}^{\infty} \beta^t \left( (\log c_t - An_t) \right. \right.
\]

\[
- \lambda_t \left( c_t + k_t - \bar{\gamma} e^{z_t} k_t^{\theta} n_t^{1-\theta} - (1 - \delta) k_{t-1} \right) \left. \right) \right]
\]
Find the first-order necessary conditions...

Differentiate:

\[
\frac{\partial L}{\partial c_t} : \quad \frac{1}{c_t} = \lambda_t
\]

\[
\frac{\partial L}{\partial n_t} : \quad A = \lambda_t (1 - \theta) \frac{y_t}{n_t}
\]

\[
\frac{\partial L}{\partial \lambda_t} : \quad c_t + k_t = \tilde{\gamma} e^{z_t} k_{t-1}^{\theta} n_t^{1-\theta} + (1 - \delta) k_{t-1}
\]

\[
\frac{\partial L}{\partial k_t} : \quad \lambda_t = \beta E_t [\lambda_{t+1} R_{t+1}]
\]

The last equation needs explanation.
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Differentiating with respect to $k_t$

- Write out the objective at date $t$: for the future, one can only form conditional expectations $E_t[\cdot]$. ’’Telescope’’ out the Lagrangian:

$$L = \ldots + \beta^t ((\log c_t - An_t)$$
$$- \lambda_t \left( c_t + k_t - \gamma e^{z_t} k_{t-1}^{\theta} n_{t-1}^{1-\theta} - (1 - \delta) k_{t-1} \right)$$
$$+ E_t \left[ \beta^{t+1} ((\log c_{t+1} - An_{t+1})$$
$$- \lambda_{t+1} \left( c_{t+1} + k_{t+1} - \gamma e^{z_{t+1}} k_t^{\theta} n_{t+1}^{1-\theta} - (1 - \delta) k_t \right) \right] + \ldots$$

- Differentiate with respect to $k_t$:

$$0 = \beta^t \lambda_t - E_t \left[ \beta^{t+1} \lambda_{t+1} \left( \theta \frac{y_{t+1}}{k_t} + 1 - \delta \right) \right]$$
Differentiating with respect to $k_t$

Sort terms and use

$$R_{t+1} = \theta \frac{y_{t+1}}{k_t} + 1 - \delta$$

to find

$$\lambda_t = \beta E_t[\lambda_{t+1} R_{t+1}]$$

This equation is called an **Euler equation** and also the Lucas asset pricing equation.
Collecting equations

1. First order conditions and a definition:

\[ \frac{1}{c_t} = \lambda_t \]

\[ A = \lambda_t (1 - \theta) \frac{y_t}{n_t} \]

\[ R_t = \theta \frac{y_t}{k_{t-1}} + 1 - \delta \]

\[ \lambda_t = \beta E_t [\lambda_{t+1} R_{t+1}] \]

2. Technology and Feasibility constraints:

\[ y_t = \bar{\gamma} e^{z_t} k_{t-1}^\theta n_t^{1-\theta} \]

\[ c_t + k_t = y_t + (1 - \delta) k_{t-1} \]

\[ z_t = \rho z_{t-1} + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma^2) \text{ i.i.d.} \]
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Step 2: Calculate the steady state

At the steady state, all variables are constant.
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Take all equations...

First order conditions and a definition:

1. \( \frac{1}{c_t} = \lambda_t \)
2. \( A = \lambda_t (1 - \theta) \frac{y_t}{n_t} \)
3. \( R_t = \theta \frac{y_t}{k_{t-1}} + 1 - \delta \)
4. \( \lambda_t = \beta E_t [\lambda_{t+1} R_{t+1}] \)

Technology and Feasibility constraints:

1. \( y_t = \tilde{\gamma} e^z t k_{t-1}^\theta n_t^{1-\theta} \)
2. \( c_t + k_t = y_t + (1 - \delta) k_{t-1} \)
3. \( Z_t = \rho Z_{t-1} + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma^2) \) i.i.d.
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... and drop the time subscripts.

1. First order conditions and a definition:

\[
\begin{align*}
\frac{1}{\bar{c}} &= \bar{\lambda} \\
\bar{A} &= \bar{\lambda}(1 - \theta) \frac{\bar{y}}{\bar{n}} \\
\bar{R} &= \theta \frac{\bar{y}}{\bar{k}} + 1 - \delta \\
\bar{\lambda} &= \beta \bar{\lambda} \bar{R}
\end{align*}
\]

2. Technology and Feasibility constraints:

\[
\begin{align*}
\bar{y} &= \bar{\gamma} e^{\bar{z}} \bar{k}^\theta \bar{n}^{1-\theta} \\
\bar{c} + \bar{k} &= \bar{y} + (1 - \delta) \bar{k} \\
\bar{z} &= \rho \bar{z}
\end{align*}
\]
Parameters

1. Calibration: \( \theta = 0.4, \, \delta = 0.012, \, \rho = 0.95, \, \sigma_\epsilon = 0.007, \)
\( \beta = 0.987, \, \gamma = 1, \) and so that \( \bar{n} = 1/3 \) (see Cooley, Frontiers...).

2. Estimation:

With numbers for the parameters, the steady state can be calculated explicitly.
Explicit calculation

From the production function,

$$\bar{y} = \bar{\gamma} e^{\bar{z}} \bar{k}^\theta \bar{n}^{1-\theta}$$

we get

$$\bar{y} = \left( \bar{\gamma} e^{\bar{z}} \left( \frac{\bar{y}}{\bar{k}} \right)^{-\theta} \right)^{\frac{1}{1-\theta}} \bar{n}$$
Explicit calculation: $\bar{n}$ given, solve for $A$.

1. $\bar{R} = \frac{1}{\beta}$

2. $\frac{\bar{y}}{\bar{k}} = \frac{\bar{R} - 1 + \delta}{\theta}$

3. $\bar{y} = \left( \gamma e^{\bar{z}} \left( \frac{\bar{y}}{\bar{k}} \right)^{-\theta} \right)^{\frac{1}{1-\theta}} \bar{n}$

4. $\bar{k} = \left( \frac{\bar{y}}{\bar{k}} \right)^{-1} \bar{y}$

5. $\bar{c} = \bar{y} - \delta \bar{k}$

6. $\bar{\lambda} = \frac{1}{\bar{c}}$

7. $A = \bar{\lambda} (1 - \theta) \frac{\bar{y}}{\bar{n}}$
Explicit calculation alternative: $A$ given, solve for $\bar{n}$.

1. $\bar{R} = \frac{1}{\beta}$

2. $\bar{y} = \bar{R} - 1 + \delta \frac{1}{\theta}$

3. $\bar{y} = \left( \bar{\gamma} e^{\bar{z}} \left( \frac{\bar{y}}{\bar{k}} \right)^{-\theta} \right)^{\frac{1}{1-\theta}}$

4. $\bar{\lambda} = \frac{A}{(1-\theta)\left( \frac{\bar{y}}{\bar{n}} \right)}$

5. $\frac{\bar{c}}{\bar{k}} = \frac{\bar{y}}{\bar{k}} - \delta$

6. $\bar{c} = \frac{1}{\bar{\lambda}}$

7. $\bar{k} = \frac{\bar{c}}{\bar{c} \bar{k}}$

8. $\bar{y} = \left( \frac{\bar{y}}{\bar{k}} \right) \bar{k}$
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Step 3: Loglinearize around the steady state

- Replace the dynamic **nonlinear** equations by dynamic **linear** equations.
- Interpretation and calculation are made easier, if the equations are linear in **percent deviations** from the steady state.
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The Principle of Loglinearization

For \( x \approx 0 \),
\[
e^x \approx 1 + x
\]

For \( x_t \), let \( \hat{x}_t = \log(x_t/\bar{x}) \) be the log-deviation of \( x_t \) from its steady state. Thus, \( 100 \times \hat{x}_t \) is (approximately) the percent deviation of \( x_t \) from \( \bar{x} \). Then,
\[
x_t = \bar{x} e^{\hat{x}_t} \approx \bar{x}(1 + \hat{x}_t)
\]
Application: The equation

\[ x_t + c_t = y_t \]

together with its steady state version

\[ \bar{x} + \bar{c} = \bar{y} \]

deliver the dynamic relationship

\[ \bar{x}\hat{x}_t + \bar{c}\hat{c}_t = \bar{y}\hat{y}_t \]
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Example: RBC

Do it slowly for two equations:

- The resource constraint:

\[
\begin{align*}
    c_t + k_t &= y_t + (1 - \delta)k_{t-1} \\
    \bar{c}e^{\hat{c}_t} + \bar{k}e^{\hat{k}_t} &= \bar{y}e^{\hat{y}_t} + (1 - \delta)\bar{k}e^{\hat{k}_{t-1}} \\
    \bar{c}(1 + \hat{c}_t) + \bar{k}(1 + \hat{k}_t) &\approx \bar{y}(1 + \hat{y}_t) + (1 - \delta)\bar{k}(1 + \hat{k}_{t-1}) \\
    (\text{Note: } \bar{c} + \delta\bar{k} &= \bar{y}) \\
    \bar{c}\hat{c}_t + \bar{k}\hat{k}_t &\approx \bar{y}\hat{y}_t + (1 - \delta)\bar{k}\hat{k}_{t-1}
\end{align*}
\]
Example: RBC

- The asset pricing equation:

\[
\lambda_t = \beta E_t [\lambda_{t+1} R_{t+1}]
\]

\[
\bar{\lambda} e^{\hat{\lambda}_t} = \beta E_t \left[ \bar{\lambda} \bar{R} e^{\hat{\lambda}_{t+1} + \hat{R}_{t+1}} \right]
\]

\[
1 + \hat{\lambda}_t \approx \beta \bar{R} E_t \left[ 1 + \hat{\lambda}_{t+1} + \hat{R}_{t+1} \right]
\]

(Note: \(1 = \beta \bar{R}\))

\[
\hat{\lambda}_t \approx E_t \left[ \hat{\lambda}_{t+1} + \hat{R}_{t+1} \right]
\]

- On “ignored” Jensen terms: can also assume joint normality of logdeviations instead. This changes the **steady state**, not the **dynamics**.
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All loglinearized equations

<table>
<thead>
<tr>
<th>#</th>
<th>Equation</th>
<th>Loglinearized</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>$\frac{1}{c_t} = \lambda_t$</td>
<td>$0 = \hat{c}_t + \hat{\lambda}_t$</td>
</tr>
<tr>
<td>(ii)</td>
<td>$A = \lambda_t(1 - \theta) \frac{y_t}{n_t}$</td>
<td>$0 = \hat{\lambda}_t + \hat{y}_t - \hat{n}_t$</td>
</tr>
<tr>
<td>(iii)</td>
<td>$R_t = \theta \frac{y_t}{k_{t-1}} + 1 - \delta$</td>
<td>$0 = -\bar{R}\hat{R}_t + \theta \frac{\bar{y}}{\bar{k}} (\hat{y}<em>t - \hat{k}</em>{t-1})$</td>
</tr>
<tr>
<td>(iv)</td>
<td>$y_t = \bar{\gamma} e^{zt} k_{t-1}^{\theta} n_t^{1-\theta}$</td>
<td>$0 = -\hat{y}<em>t + z_t + \theta \hat{k}</em>{t-1} + (1 - \theta) \hat{n}_t$</td>
</tr>
<tr>
<td>(v)</td>
<td>$c_t + k_t = y_t + (1 - \delta) k_{t-1}$</td>
<td>$0 = -\bar{c} \hat{c}_t - \bar{k} \hat{k}_t + \bar{y} \hat{y}<em>t + (1 - \delta) \bar{k} \hat{k}</em>{t-1}$</td>
</tr>
<tr>
<td>(vi)</td>
<td>$\lambda_t = \beta E_t[\lambda_{t+1} R_{t+1}]$</td>
<td>$0 = -\hat{\lambda}<em>t + E_t[\hat{\lambda}</em>{t+1} + \hat{R}_{t+1}]$</td>
</tr>
<tr>
<td>(vii)</td>
<td>$Z_{t+1} = \rho Z_t + \epsilon_{t+1}$</td>
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Recursivity

- State variables are: $k_{t-1}, z_t$ (or, alternatively, $k_{t-1}$ and $z_{t-1}$).
- The dynamics of the model should be describable by a recursive law of motion (RLOM),

$$
\begin{align*}
\lambda_t &= f(\lambda)(k_{t-1}, z_t) \\
k_t &= f(k)(k_{t-1}, z_t) \\
y_t &= f(y)(k_{t-1}, z_t)
\end{align*}
$$

etc.
Assume that the RLOM is linear in the log-deviations,

\[ \hat{\lambda}_t = \eta_{\lambda k} \hat{k}_{t-1} + \eta_{\lambda z} z_t \]
\[ \hat{k}_t = \eta_{kk} \hat{k}_{t-1} + \eta_{kz} z_t \]
\[ \hat{y}_t = \eta_{yk} \hat{k}_{t-1} + \eta_{yz} z_t \]

etc. for coefficients \( \eta_{\lambda k}, \eta_{\lambda z}, \) etc.

To make life simpler here, we shall try to reduce the system to only \( k \) and \( \lambda \) (one doesn’t have to).
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Simplify:

• Note: 
\[ \hat{y}_t = \frac{1}{\theta} z_t + \hat{k}_{t-1} + \frac{1-\theta}{\theta} \hat{\lambda}_t \]

• Abbreviations:

\[ \alpha_1 = \frac{\bar{y}}{\bar{k}} + (1 - \delta) \]
\[ \alpha_2 = \frac{\bar{c}}{\bar{k}} + \frac{1 - \theta}{\theta} \frac{\bar{y}}{\bar{k}} \]
\[ \alpha_3 = \frac{\bar{y}}{\theta \bar{k}} \]
\[ \alpha_4 = 0 \]
\[ \alpha_5 = 1 + (1 - \theta) \frac{\bar{y}}{\bar{R} \bar{k}} \]
\[ \alpha_6 = \frac{\bar{y}}{\bar{R} \bar{k}} \]
Obtaining the solution

We obtain the following \textbf{first-order two-dimensional stochastic difference equation}:

\begin{align*}
0 &= -\hat{k}_t + \alpha_1 \hat{k}_{t-1} + \alpha_2 \hat{\lambda}_t + \alpha_3 Z_t \\
0 &= E_t[-\hat{\lambda}_t + \alpha_4 k_t + \alpha_5 \hat{\lambda}_{t+1} + \alpha_6 Z_{t+1}] \\
Z_t &= \rho Z_{t-1} + \epsilon_t
\end{align*}

where $Z_t$ is an exogenous stochastic process.
Obtaining the solution

- Compare to the following first-order two-dimensional stochastic difference equation to be studied in the lecture on difference equations:

\[
0 = -x_t + \alpha_1 x_{t-1} + \alpha_2 y_t + \alpha_3 z_t 
\]  
\[
0 = E_t[-y_t + \alpha_4 x_t + \alpha_5 y_{t+1} + \alpha_6 z_{t+1}] 
\]  
\[
z_t = \rho z_{t-1} + \epsilon_t 
\]

They are the same with \( x_t = \hat{k}_t, y_t = \hat{\lambda}_t \).
The Method of Undetermined Coefficients

Postulate the \textbf{recursive law of motion}

\begin{align*}
\hat{\lambda}_t &= \eta_{\lambda k} \hat{k}_{t-1} + \eta_{\lambda z} z_t \tag{7} \\
\hat{k}_t &= \eta_{kk} \hat{k}_{t-1} + \eta_{kz} z_t \tag{8}
\end{align*}

Plug this into equations (1) once and (2) “twice” and exploit $E_t[z_{t+1}] = \rho z_t$, so that \textbf{only the date-t-states $\hat{k}_{t-1}$ and $z_t$ remain},

\begin{align*}
0 &= (-\eta_{kk} + \alpha_1 + \alpha_2 \eta_{\lambda k}) \hat{k}_{t-1} \\
&+ (-\eta_{kz} + \alpha_2 \eta_{\lambda z} + \alpha_3) z_t \\
0 &= (-\eta_{\lambda k} + \alpha_4 \eta_{kk} + \alpha_5 \eta_{\lambda k} \eta_{kk}) \hat{k}_{t-1} \\
&+ (-\eta_{\lambda z} + \alpha_4 \eta_{kz} + \alpha_5 \eta_{\lambda k} \eta_{kz} + (\alpha_5 \eta_{\lambda z} + \alpha_6) \rho) z_t
\end{align*}

\textbf{Compare coefficients}
On plugging in twice...

Plugging

\[
\hat{\lambda}_t = \eta_{\lambda k} \hat{k}_{t-1} + \eta_{\lambda z} z_t, \quad \hat{k}_t = \eta_{kk} \hat{k}_{t-1} + \eta_{kz} z_t \quad \rho z_t = E_t[z_{t+1}]
\]

twice into

\[
0 = E_t[-\hat{\lambda}_t + \alpha_4 \hat{k}_t + \alpha_5 \hat{\lambda}_{t+1} + \alpha_6 z_{t+1}]
\]

\[
= E_t[-(\eta_{\lambda k} \hat{k}_{t-1} + \eta_{\lambda z} z_t) + \alpha_4 (\eta_{kk} \hat{k}_{t-1} + \eta_{kz} z_t) + \alpha_5 (\eta_{\lambda k} \hat{k}_{t} + \eta_{\lambda z} z_{t+1}) + \alpha_6 z_{t+1}]
\]

\[
= E_t[-\eta_{\lambda k} \hat{k}_{t-1} - \eta_{\lambda z} z_t + \alpha_4 \eta_{kk} \hat{k}_{t-1} + \alpha_4 \eta_{kz} z_t + \alpha_5 \eta_{\lambda k} (\eta_{kk} \hat{k}_{t-1} + \eta_{kz} z_t) + (\alpha_5 \eta_{\lambda z} + \alpha_6) z_{t+1}]
\]

\[
= \left(-\eta_{\lambda k} + \alpha_4 \eta_{kk} + \alpha_5 \eta_{\lambda k} \eta_{kk}\right) \hat{k}_{t-1}
\]

\[
+ \left(-\eta_{\lambda z} + \alpha_4 \eta_{kz} + \alpha_5 \eta_{\lambda k} \eta_{kz} + (\alpha_5 \eta_{\lambda z} + \alpha_6) \rho\right) z_t
\]
Comparing coefficients

On $\hat{k}_{t-1}$:

\[
0 = -\eta_{kk} + \alpha_1 + \alpha_2 \eta_{\lambda k} \\
0 = -\eta_{\lambda k} + \alpha_4 \eta_{kk} + \alpha_5 \eta_{\lambda k} \eta_{kk}
\]

One gets the characteristic quadratic equation

\[
0 = p(\eta_{kk}) = \eta_{kk}^2 - \left( \alpha_1 - \frac{\alpha_2}{\alpha_5} \alpha_4 + \frac{1}{\alpha_5} \right) \eta_{kk} + \frac{\alpha_1}{\alpha_5}
\] (9)
Solving the characteristic equation

Solutions:

\[ \eta_{kk} = \frac{1}{2} \left( \left( \alpha_1 - \frac{\alpha_2}{\alpha_5} \alpha_4 + \frac{1}{\alpha_5} \right) \pm \sqrt{\left( \alpha_1 - \frac{\alpha_2}{\alpha_5} \alpha_4 + \frac{1}{\alpha_5} \right)^2 - 4 \frac{\alpha_1}{\alpha_5}} \right) \]

Choose the stable root \( |\eta_{kk}| < 1 \). There is at most one stable root, if

\[ |\eta_{kk,1}| \eta_{kk,2} = \left| \frac{\alpha_1}{\alpha_5} \right| > 1 \]

With \( \eta_{kk} \), calculate

\[ \eta_{\lambda k} = \frac{\eta_{\lambda \lambda} - \alpha_1}{\alpha_2} \]
Comparing coefficients

On $z_t$:

\[ 0 = -\eta_k z + \alpha_2 \eta_{\lambda z} + \alpha_3 \]
\[ 0 = -\eta_{\lambda z} + \alpha_4 \eta_k z + \alpha_5 \eta_{\lambda k} \eta_{kz} + \left( \alpha_5 \eta_{\lambda z} + \alpha_6 \right) \rho \]

Solution:

\[ \eta_{\lambda z} = \frac{\alpha_4 \alpha_3 + \alpha_5 \eta_{\lambda k} \alpha_3 + \alpha_6 \rho}{1 - \alpha_4 \alpha_2 - \alpha_5 \eta_{\lambda k} \alpha_2 - \alpha_5 \rho} \]
\[ \eta_{kz} = \alpha_2 \eta_{\lambda z} + \alpha_3 \]
The solution strategy

Hansens benchmark Real Business Cycle Model

The solution steps

Step 1: find the FONCs
Step 2: Calculate the steady state
Step 3: Loglinearize
Step 4: Solve for the RLOM
Step 5: Calculate impulse responses

Representations

Alternative representations
Step 5: Calculate impulse responses and (HP-filtered) moments

- Impulse responses: will be explained now.
- HP-filtered moments: will be discussed later.
Impulse Response Functions: response to a shock in $z_t$

1. Set $z_0 = 0$, $\epsilon_1 = 1$, $\epsilon_t = 0$, $t > 1$
2. Calculate $z_t = \rho^t$
3. Set $\hat{k}_0 = 0$.
4. Calculate recursively

$$\hat{k}_t = \eta_{kk}\hat{k}_{t-1} + \eta_{kz}z_t$$

5. With that, calculate

$$\hat{\lambda}_t = \eta_{\lambda k}\hat{k}_{t-1} + \eta_{\lambda z}z_t$$
The solution strategy
Hansen's benchmark Real Business Cycle Model

The solution steps
1. Find the FONCs
2. Calculate the steady state
3. Loglinearize
4. Solve for the RLOM
5. Calculate impulse responses

Results: Impulse Responses to shocks

Impulse responses to a shock in technology

<table>
<thead>
<tr>
<th>Years after shock</th>
<th>Percent deviation from steady state</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>capital</td>
</tr>
<tr>
<td></td>
<td>consumption</td>
</tr>
<tr>
<td></td>
<td>labor</td>
</tr>
<tr>
<td></td>
<td>output</td>
</tr>
<tr>
<td></td>
<td>technology</td>
</tr>
<tr>
<td></td>
<td>interest</td>
</tr>
</tbody>
</table>

Impulse responses to a shock in technology graph:
- Output increases significantly after the shock.
- Consumption follows the output pattern, but with a lag.
- Capital and interest are less responsive but show a rise.
- Labor and technology show a decrease initially and then stabilize.
Impulse Response Functions: response to an initial deviation of the state $k_t$ from its steady state.

1. Set $z_t = 0$, $t \geq 1$
2. Set $\hat{k}_0 = 1$.
3. Calculate recursively

$$\hat{k}_t = \eta_{kk} \hat{k}_{t-1}$$

4. With that, calculate

$$\hat{\lambda}_t = \eta_{\lambda k} \hat{k}_{t-1}$$
The solution strategy
Hansens benchmark Real Business Cycle Model

The solution steps
Step 1: find the FONCs
Step 2: Calculate the steady state
Step 3: Loglinearize
Step 4: Solve for the RLOM
Step 5: Calculate impulse responses

Results: Impulse Responses to capital deviations

Prof. H. Uhlig  IAMA: Lecture 5
Outline

1. The solution strategy
   - Overview

2. Hansens benchmark Real Business Cycle Model
   - The model
   - Rational expectations
   - Labor supply

3. The solution steps
   - Step 1: find the FONCs
   - Step 2: Calculate the steady state
   - Step 3: Loglinearize
   - Step 4: Solve for the RLOM
   - Step 5: Calculate impulse responses

4. Representations
   - Alternative representations
# Recall: the loglinearized equations

<table>
<thead>
<tr>
<th>#</th>
<th>Equation</th>
<th>Loglinearized</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>$\frac{1}{c_t} = \lambda_t$</td>
<td>$0 = \hat{c}_t + \hat{\lambda}_t$</td>
</tr>
<tr>
<td>(ii)</td>
<td>$A = \lambda_t (1 - \theta) \frac{y_t}{n_t}$</td>
<td>$0 = \hat{\lambda}_t + \hat{y}_t - \hat{n}_t$</td>
</tr>
<tr>
<td>(iii)</td>
<td>$R_t = \theta \frac{y_t}{k_{t-1}} + 1 - \delta$</td>
<td>$0 = -\bar{R}R_t + \theta \bar{y} \left( \hat{y}<em>t - \hat{k}</em>{t-1} \right)$</td>
</tr>
<tr>
<td>(iv)</td>
<td>$y_t = \bar{\gamma} e^{zt} k_{t-1}^{\theta} n_t^{1-\theta}$</td>
<td>$0 = -\hat{y}<em>t + z_t + \theta \hat{k}</em>{t-1} + (1 - \theta) \hat{n}_t$</td>
</tr>
<tr>
<td>(v)</td>
<td>$c_t + k_t = y_t + (1 - \delta) k_{t-1}$</td>
<td>$0 = -\bar{c} \hat{c}_t - \bar{k} \hat{k}_t + \bar{y} \hat{y}<em>t + (1 - \delta) \bar{k} \hat{k}</em>{t-1}$</td>
</tr>
<tr>
<td>(vi)</td>
<td>$\lambda_t = \beta E_t[\lambda_{t+1} R_{t+1}]$</td>
<td>$0 = -\hat{\lambda}<em>t + E_t[\hat{\lambda}</em>{t+1} + \hat{R}_{t+1}]$</td>
</tr>
<tr>
<td>(vii)</td>
<td>$Z_{t+1} = \rho Z_t + \epsilon_{t+1}$</td>
<td>$Z_{t+1} = \rho Z_t + \epsilon_{t+1}$</td>
</tr>
</tbody>
</table>
A representation of the problem

There is an endogenous state vector $x_t$, size $m \times 1$, a list of other endogenous variables $y_t$, size $n \times 1$, and a list of exogenous stochastic processes $z_t$, size $k \times 1$. The equilibrium relationships between these variables are

\[
egin{aligned}
0 &= Ax_t + Bx_{t-1} + Cy_t + Dz_t \\
0 &= E_t[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t] \\
z_{t+1} &= Nz_t + \epsilon_{t+1}; \quad E_t[\epsilon_{t+1}] = 0,
\end{aligned}
\]

where it is assumed that $C$ is of size $l \times n$, $l \geq n$ and of rank $n$, that $F$ is of size $(m + n - l) \times n$, and that $N$ has only stable eigenvalues.
Example: RBC

Variables:

\[ x_t = [\text{capital}] = [\hat{k}_t], \quad y_t = \begin{bmatrix} \text{Lagrangian} \\ \text{consumption} \\ \text{output} \\ \text{labor} \\ \text{interest} \end{bmatrix} = \begin{bmatrix} \hat{\lambda}_t \\ \hat{c}_t \\ \hat{y}_t \\ \hat{n}_t \\ \hat{R}_t \end{bmatrix} \]

and

\[ z_t = [\text{technology}] = [z_t] \]
The solution strategy
Hansens benchmark Real Business Cycle Model
The solution steps
Representations

Example: RBC

Matrices:

\[
A = \begin{bmatrix}
  0 \\
  0 \\
  0 \\
  -\bar{k}
\end{bmatrix},
B = \begin{bmatrix}
  0 \\
  0 \\
  -\theta \bar{y} \\
  (1 - \delta)\bar{k}
\end{bmatrix},
C = \begin{bmatrix}
  1 & 1 & 0 & 0 & 0 \\
  1 & 0 & 1 & -1 & 0 \\
  0 & 0 & \theta \bar{y} & 0 & -\bar{R} \\
  0 & 0 & -1 & (1 - \theta) & 0 \\
  0 & -\bar{c} & \bar{y} & 0 & 0
\end{bmatrix}
\]

and

\[
F = [0],
G = [0],
H = [0],
J = [1, 0, 0, 0, 1],
K = [-1, 0, 0, 0, 0],
L = [0],
M = [0],
N = [\rho]
\]
Redefine the system as

\[
\begin{align*}
\tilde{x}_t &= \begin{bmatrix} x_t \\ y_t \end{bmatrix}, \\
\tilde{F} &= \begin{bmatrix} 0 & 0 \\ F & J \end{bmatrix}, \\
\tilde{G} &= \begin{bmatrix} A & C \\ G & K \end{bmatrix}, \\
\tilde{H} &= \begin{bmatrix} B & 0 \\ H & 0 \end{bmatrix}, \\
\tilde{L} &= \begin{bmatrix} 0 \\ L \end{bmatrix}, \\
\tilde{M} &= \begin{bmatrix} D \\ M \end{bmatrix},
\end{align*}
\]

The system can then be rewritten as a second-order stochastic matrix difference equation,

\[
0 = E_t \left[ F\tilde{x}_{t+1} + \tilde{G}\tilde{x}_t + \tilde{H}\tilde{x}_{t-1} + \tilde{L}z_{t+1} + \tilde{M}z_t \right]
\]

\[
z_t = Nz_{t-1} + \epsilon_t; \quad E_{t-1}[\epsilon_t] = 0 + \tilde{\epsilon}_t
\]
The solution strategy
Hansens benchmark Real Business Cycle Model
The solution steps
Redefine the system as
\[
\tilde{x}_t = \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix}, \quad \tilde{\varepsilon}_t = \begin{bmatrix} 0 \\ 0 \\ \varepsilon_t \end{bmatrix},
\]
\[
\tilde{F} = \begin{bmatrix} 0 & 0 & 0 \\ F & J & L \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{G} = \begin{bmatrix} A & C & D \\ G & K & M \\ 0 & 0 & -I_k \end{bmatrix}, \quad \tilde{H} = \begin{bmatrix} B & 0 & 0 \\ H & 0 & 0 \\ 0 & 0 & N \end{bmatrix}
\]
The system can then be rewritten as a second-order stochastic matrix difference equation,
\[
0 = E_t \left[ F\tilde{x}_{t+1} + \tilde{G}\tilde{x}_t + \tilde{H}\tilde{x}_{t-1} \right] + \tilde{\varepsilon}_t
Alternative Representations 3

- E.g. per stacking,

\[ \tilde{x}_t = \begin{bmatrix} \tilde{x}_t \\ \tilde{x}_{t-1} \end{bmatrix} \]

etc., one can even rewrite the system as a first-order stochastic matrix difference equation,

\[ 0 = E_t \left[ F \tilde{x}_{t+1} + G \tilde{x}_t \right] + \tilde{\epsilon}_t \]

- Here, one needs to keep in mind, that some entries in \( x_t \) are predetermined, i.e. already fixed as of \( t - 1 \).

- This representation is often used, e.g. in Blanchard-Kahn, Farmer, many others.
Various representations appear in the literature.
Which representation is most convenient? That depends on the solution approach.
The “complicated” first representation has the advantage of focussing on a small number of state variables.