Appendix to

Technological Opportunity, Long-Run Growth, and Convergence[1]

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1 Evidence Supporting the Setup

There exists a large amount of anecdotal evidence which can be used to support the view that technological opportunity is a relevant concept and that the distinction between incremental and radical innovations is helpful for the proper understanding of technological change across centuries (cf. Olsson [8]). Let us present a few illustrative examples.

Example 1. In 1814, Joseph Niépce invented the first photo camera. It took 8 hours to take one picture. This certainly must be viewed as a radical innovation but it was not yet a useful technology for increasing the productivity of the economy. However, in 1851 the exposure time was reduced to 2-3 seconds, in 1888 the first roll-film was developed, and in 1941 – the color film. One and a half century later, we have digital cameras which are available at prices accessible to the general public and pictures can be printed at home. It is clear that the small improvements to the photo camera, in our terminology the incremental innovations, ought to be viewed as the crucial steps for spreading the technology into the economy,

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whereas the radical innovation of Joseph Niépce was the one which opened up the opportunity for these incremental innovations.

Example 2. The first locomotive was developed in 1804 by Richard Trevithick. It was the first steam-powered locomotive, and therefore ought to be considered a radical innovation. However, it was too heavy and even broke the very own rails it was supposed to travel on. Compared to this, the incremental innovations following that radical innovation were tremendous. In 1814 came the first steam locomotive that was actually able to travel, although at only 6 km/h; today the Maglev, the high-speed magnetic train, travels at more than 550 km/h. Again, the initial idea of Richard Trevithick was the one opening up the possibilities for the incremental innovations, whereas the radical innovation proved useless for improving productivity.

Example 3. In 1928, Alexander Fleming had, by accident, left a Staphylococcus plate culture lying in the warm cellar. Several days later, upon reminding himself of the forgotten plate culture, he noticed that there was a blue-green colored mold destroying the bacteria. He called this mold Penicillin. It was however too weak and unstable to provide a useful means of destroying bacterial infections in humans. Only subsequent research by Chain, Florey and Heatley developed the kind of Penicillin which now saves human lives throughout the world. As Sir Henry Harris had aptly put it: "Without Fleming, no Chain; without Chain, no Florey; without Florey, no Heatley; without Heatley, no Penicillin."

Similar stories can be told about the invention of the first battery by Alessandro Volta in 1799, the first champagne by Dom Pérignon in 1670, nylon by DuPont in 1928, the steam engine, the airplane, electricity, and many more. What can be seen here is that we view radical innovations as spanning a broader class of innovations than for example General Purpose Technologies. They all have one thing in common: opening up opportunities for small improvements, for incremental innovations that help make the technology accessible, practical and operative. Without the radical innovations being able to open up new opportunities, there would be no place for incremental innovations. And without doubt, in a considerable if not exhaustive number of cases, it were the incremental innovations which really proved to be useful for economic purposes.

A radical innovation is usually the first one in a series of innovations. It opens up a new line of research. It should not be mistaken, however, with an *early* innovation: for example, the early invention of gunpowder in ancient China did not open up many opportunities for developments, while the invention of MP3 music encoding did so despite being quite recent.

Our understanding of the evolution of technological knowledge differs from the articles basing on combinatorial calculations (Romer [10]; Weitzman [11]). We do not consider the potential for technological change as the number of possible ways to combine ideas. For example, 20 objects may be combined in $2^{20} = 1,048,576$ ways. Given an enormous number like this, papers in this vein conclude that there are no practical limits to technological change. However, why should we combine every possible idea? Intuitively, it seems more likely that just the "non-dominated" technologies, those on the technological frontier (Caselli and Coleman [2]), are improved. As Poincaré observed: "To create consists precisely in not making useless combinations." So, we suggest that only ideas on the technology frontier may be usefully improved, an assumption which this article shares with Kortum [6] and Olsson [8]. The implication is that incremental innovations would by themselves come to a halt if the technological frontier were not constantly pushed ahead by radical innovations.

So, how does technological opportunity increase? What qualifies as a radical innovation? We suggest that a radical innovation arises whenever a new class of known phenomena (physical, biological, chemical, etc.) is found to have economically useful applications. For example, "[f]or thousands of years, silicon dioxide provided utility mainly as sand on the beach, but now it delivers utility through the myriad of goods that depend on computer chips" (Jones [4]). Conclusively, we believe that the flows of radical innovations are related to the stock of existing knowledge. The larger the existing knowledge base, the more discoveries will be transformed into radical innovations.

2 Technological Opportunity and the Evolution of Knowledge

2.1 The Relationship Between Inventions

This section demonstrates our version of the set-theoretic approach to modeling ideas put forward by Olsson [7], [8]. This approach should be understood as a metaphor of the evolution of human thought in reality. Consequently, all the concepts defined precisely in this metaphorical world (such as technological opportunity) are supposed to have real-world analogues. Figure 1 illustrates the set-theoretic approach in a two-dimensional case.

We start with the comforting assumption that any kind of idea can potentially be utilized. The space of ideas (known and unknown) is the whole $\mathbb{R}^n$ space, where $n \in \mathbb{N}$ reflects the (large) number of dimensions in which ideas are characterized. In this world of ideas, the standard Pythagorean distance metric $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is assumed to apply: $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$. We use the Lebesgue measure λ to describe the volume of all Lebesgue measurable subsets of $\mathbb{R}^n$.

The **technology set** $\mathcal{A} \subset \mathbb{R}^n$ is assumed to be a compact and connected set encompassing all ideas that are already known. We write $\lambda(\mathcal{A}) = A$. The convex hull of the technology set, $\mathcal{P} \supset \mathcal{A}$, typically consisting of both some known and some unknown ideas, will be called the technological **paradigm**.

Within the technology set, there is the set of ignorance, called **0** $\in \mathcal{A}$, which can be thought of as consisting of some natural ability and instinct, devoid of any systematic knowledge. The set of ignorance has a small but nevertheless positive measure, $\lambda(0) > 0$. It is assumed that in the beginning of time, the technology set $\mathcal{A} \subset \mathbb{R}^n$ is comprised of the set **0** only.

Discoveries form a set $\mathcal{D} = \{d_1, d_2, ..., d_m\}$, $\mathcal{D} \in \mathbb{R}^n \backslash \mathrm{co}(\mathcal{A})$ with $m \in \mathbb{N}$, of isolated points away from the current paradigm (i.e., the convex hull of the current technology set). At this point, we choose to link the amount of discoveries made at a given moment in time to the current technology level A – i.e. to the current measure of the technology set. In principle, one should allow either for increasing,

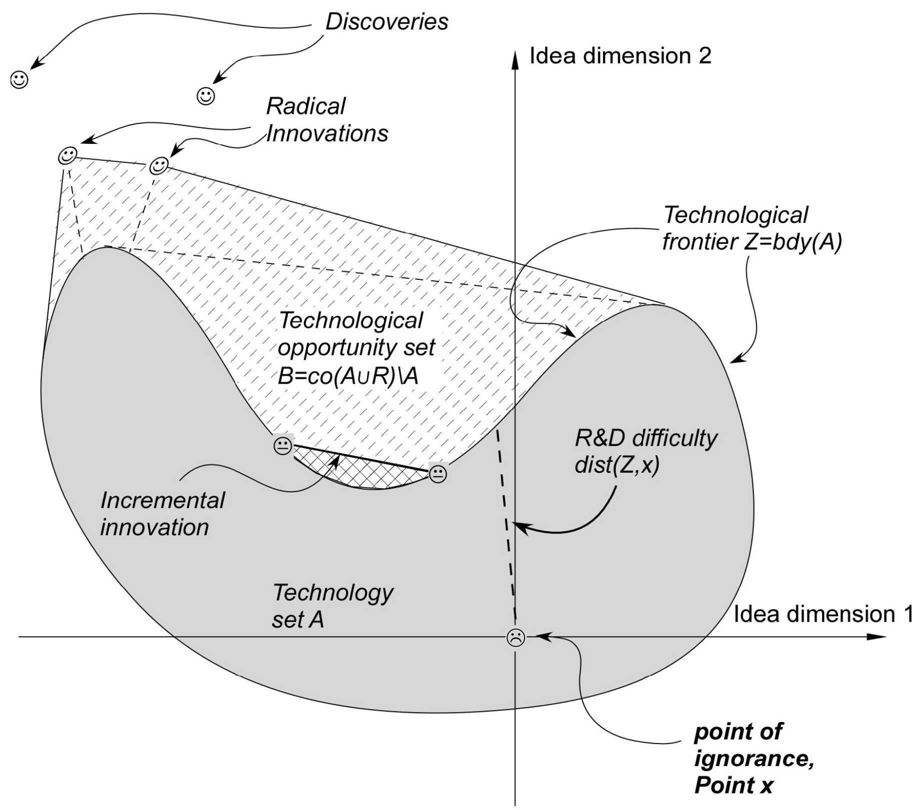

Figure 1: The set-theoretic approach illustrated.

constant, or decreasing returns to scale here (IRTS, CRTS, DRTS, respectively). It turns out however that only the DRTS case does not lead to explosive dynamics. Therefore, we write $D \propto \#\mathcal{D}$, where $D = g(A) \geq 0$, with $g'(A) \geq 0$ having DRTS.

A **radical innovation** is one that connects points on the boundary of the technology set $\mathcal{A}$ with discoveries, and so the **radical innovation set** is given by $\mathcal{R} = \bigcup_{i=1,2,\ldots,r} I[x_i, d_i]$, $\forall i(x_i \in \text{bdy}(\mathcal{A}), d_i \in \mathcal{D})$, and $r \in \mathbb{N}$ with $r \leq m$. Please note that the measures of the radical innovation set as well as of the discoveries set are always zero, and thus radical innovations and discoveries cannot provide additions to the measure of the technology set: like in Olsson [8], they do not improve technology directly.

In the subsequent sections, we will be also interested in the measure of the convex hull of the radical innovation set. We shall write $R = \lambda(co(\mathcal{R}))$.

The possibilities for further technological developments are collected in the **technological opportunity set** $\mathcal{B} \subset \mathbb{R}^n$. Formally, this set is comprised of all ideas that belong to the paradigm (convex hull of the technology set) but do not belong to the technology set itself: $\mathcal{B} = \text{co}(\mathcal{A})\backslash\mathcal{A}$. Of course, if the technology set is convex ($\mathcal{A} = \text{co}(\mathcal{A})$), then the technological opportunity set is empty. We shall denote the extent of technological opportunity by $B = \lambda(\mathcal{B})$.

Developments combining ideas from the boundary of the technology set are grouped together in the **incremental innovation set** $\mathcal{C} \subseteq \mathcal{B}$. The inclusion $\mathcal{C} \subseteq \mathcal{B}$ means that one can work out gradual developments only if the opportunity for them already exists. As already indicated before, this is an assumption where Olsson [8] and this article clearly depart from the usual modeling practice à la Romer [9], Aghion and Howitt [1], or Jones [3].

The "incremental", gradual nature of incremental innovations is captured by assuming that the distance between two combined ideas $i_m, i_n \in \text{bdy}(\mathcal{A})$ cannot exceed some given constant $\bar{d}$: $d(i_m, i_n) \leq \bar{d}$. In addition, a new idea can only be a convex combination of already known ideas, $i_\epsilon = \alpha i_m + (1 - \alpha)i_n$ with $0 < \alpha < 1$, where the new idea incrementally expands the technology set, such that $i_\epsilon \in \text{co}(\mathcal{A})\backslash\mathcal{A}$. We denote the measure of the incremental innovation set by $C = \lambda(\mathcal{C})$.

Consistently with these assumptions, it follows that technological opportunity gets depleted by incremental innovations but renewed by radical innovations: $\dot{B} = -C + R$. The technology set is increased by incremental innovations and is not altered by radical innovations: $\dot{A} = C$.

One interesting question that remains is whether with larger technological opportunity it becomes increasingly easier to produce incremental innovations, or, on the contrary, it becomes harder to produce them. We are not aware of any empirical evidence on this, but for the sake of plausibility, we shall concentrate on the DRTS case here. Again, it is the unique case that does not lead to explosive dynamics.

When constructing our model, we shall also make the usual assumption that each individual in the population of size L splits her fixed time endowment ($1 = \ell_Y + \ell_A$) between working, ℓ_Y, searching for radical innovations, $(1-u)\ell_A$, and doing incremental R&D, $u\ell_A$. This assumption allows us to clarify the evolution of the measures of the sets as follows: $\dot{B} = -C(u\ell_A L, B) + R((1-u)\ell_A L, A)$; $\dot{A} = C(u\ell_A L, B)$.

2.2 Detailed specification of the set-theoretic approach

We shall now present our set-theoretic approach to modeling technical change in a more rigorous and compact form. This approach has been first formulated by Olsson [7] and then refined by the same author in Olsson [8]. Here we have added in more structure (Lebesgue measure, Pythagorean metric, point of ignorance, R&D difficulty).

Definition 1 *The* idea space *is the $\mathbb{R}^n$ space, where $n \in \mathbb{N}$ is predefined, together with the Lebesgue measure λ defined on all Borel subsets of $\mathbb{R}^n$, and the usual Pythagorean metric $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$.*

Definition 2 *The* technology set *$\mathcal{A} \subset \mathbb{R}^n$ is a compact and connected set such that $0 \in \mathcal{A}$.*

Please note that the origin $0 \in \mathbb{R}^n$ is labeled also the *point of ignorance*; the set of ignorance is a ball of some small diameter $\bar{\epsilon} > 0$, surrounding the point 0.

Definition 3 *The* technology level *is the Lebesgue measure of the technology set, $\lambda(\mathcal{A}) > 0$.*

Definition 4 *The* paradigm *$\mathcal{P} \subset \mathbb{R}^n$ is the convex hull of the technology set: $\mathcal{P} \equiv \mathrm{co}(\mathcal{A})$.*

The paradigm is a compact, connected and convex set. We also have that $0 \in \mathcal{P}$, $\mathcal{A} \subset \mathcal{P}$, and $\lambda(\mathcal{P}) \geq \lambda(\mathcal{A}) > 0$.

Definition 5 *The* technological opportunity set $\mathcal{B} \subset \mathbb{R}^n$ *is (the closure of) the set of ideas that belong to the paradigm but do not belong to the technology set:* $\mathcal{B} = cl(\mathcal{P} \backslash \mathcal{A})$.

The technological opportunity set is a compact set, and $0 \notin \mathcal{B}$.

Definition 6 *The level of* technological opportunity *is the measure of the technological opportunity set* $\lambda(\mathcal{B})$.

Definition 7 *The* technology frontier $\mathcal{Z}$ *is the set of ideas that belong both to the technology set and to the technological opportunity set:* $\mathcal{Z} \equiv \mathcal{A} \cap \mathcal{B}$.

The technology frontier is a compact boundary set ($int\mathcal{Z} = \emptyset, \lambda(\mathcal{Z}) = 0$).

Definition 8 *The* R&D difficulty *is the distance of the technology frontier to the origin:* $\bar{\delta} \equiv dist(\mathcal{Z}, 0)$. *If the technology frontier is empty, we put* $\bar{\delta} = dist(bdy\mathcal{A}, 0)$.

Definition 9 *The* incremental innovation set *is a subset of the technological opportunity set,* $\mathcal{C} \subset \mathcal{B}$*, with* $\mathcal{C} \cap \mathcal{Z} \neq \emptyset$ *if* $\mathcal{Z} \neq \emptyset$.

Definition 10 *The* discoveries set *is a finite set of isolated points outside the paradigm:* $\mathcal{D} = \{d_1, d_2, ..., d_m\}$*,* $\mathcal{D} \in \mathbb{R}^n \backslash \mathcal{P}$*, and* $m \in \mathbb{N}$.

The discoveries set is a compact boundary set, with $int\mathcal{D} = \emptyset, \lambda(\mathcal{D}) = 0$.

Definition 11 *The* radical innovation set *is a set of closed intervals, connecting points on the boundary of the technology set with discoveries:*
$\mathcal{R} = \bigcup_{i=1,2,...,r} I[x_i, d_i]$*, where* $\forall i(x_i \in bdy\mathcal{A}, d_i \in \mathcal{D})$*, and* $r \in \mathbb{N}$ *with* $r \leq m$.

The radical innovation set is a compact boundary set, $int\mathcal{R} = \emptyset, \lambda(\mathcal{R}) = 0$.

2.3 Evolution of the Sets Over Time

We assume that in a period of time of an infinitesimal length $\varepsilon > 0$,

$$\mathcal{A}_\varepsilon = \mathcal{A} \cup \mathcal{C} \cup \mathcal{R},$$

i.e. newly invented technologies add to the technology set. It follows that the paradigm is shifted by radical innovations; technological opportunity is diminished by incremental innovations and extended by radical innovations, and the technology frontier is pushed forward:

$$\begin{aligned}
\mathcal{P}_\varepsilon &= \mathrm{co}(\mathcal{A}_\varepsilon), \\
\mathcal{B}_\varepsilon &= cl(\mathcal{P}_\varepsilon \backslash \mathcal{A}_\varepsilon) = cl\left(\underbrace{(\mathcal{B}\backslash\mathcal{C})}_{\text{extraction}} \cup \underbrace{(\mathcal{P}_\varepsilon\backslash\mathcal{P})}_{\text{paradigm shift}}\right), \\
\mathcal{Z}_\varepsilon &= \mathcal{A}_\varepsilon \cap \mathcal{B}_\varepsilon.
\end{aligned}$$

Bearing in mind that the radical innovation set is of measure zero but its convex hull is (typically) of positive measure, we find that the technology level and the level of technological opportunity evolve in the following way:

$$\begin{aligned}
\lambda(\mathcal{A}_\varepsilon) &= \lambda(\mathcal{A}) + \lambda(\mathcal{C}), \\
\lambda(\mathcal{B}_\varepsilon) &= \lambda(\mathcal{B}) - \lambda(\mathcal{C}) + \lambda(\mathcal{P}_\varepsilon \backslash \mathcal{P}).
\end{aligned}$$

The last "paradigm shift" term is driven by radical innovations in $\mathcal{R}$ only. Please note that only incremental innovations have the power to increase the measure of the technology set.

R&D difficulty evolves as follows:

$$\bar{\delta}_\varepsilon = dist(\mathcal{Z}_\varepsilon, 0) \geq \bar{\delta}.$$

This set-theoretic approach is one way of providing a foundation of the dynamic equations of our model, summarized in the system of equations

$$\begin{aligned}
\dot{A}_t &= \delta(u_t \ell_{At} L_t)^\beta B_t^\mu, && (1) \\
\dot{B}_t &= \underbrace{-\delta(u_t \ell_{At} L_t)^\beta B_t^\mu}_{\text{incremental innovations}} + \underbrace{\gamma((1-u_t)\ell_{At} L_t)^\beta A_t^\nu}_{\text{radical innovations}}. && (2)
\end{aligned}$$

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