



Production functions and distributions of unit factor productivities: Uncovering the link

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ABSTRACT

We derive a reversible “endogenous technology choice transform,” according to which firm-level production functions and distributions of unit factor productivities are two sides of the same coin. The Cobb–Douglas function relates to Pareto distributions, and the CES to Weibull distributions.

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1. Introduction

This paper accepts Jones' (2005) view that the production function, commonly assumed by macroeconomists to be a primitive, is in fact only a reduced form which should be derived from microfoundations. The firm-level production function is considered to be an assembly of a multiplicity of production techniques, particular methods of producing the final good: each method is characterized by an n -tuple of unit factor productivities (UFPs hereafter), one per factor of production. We allow such an n -tuple to be chosen optimally by the firm from the set of all available and non-dominated n -tuples which we call the technology frontier.

It is known that the Cobb–Douglas production function can be derived as an assembly (convex hull) of production techniques if unit factor productivities are drawn from independent Pareto distributions (Kortum, 1997; Jones, 2005). In a paper complementary to this one (Growiec, in press), we have shown that allowing for dependence between these Pareto distributions enables one to obtain a wide range of possible production functions, including the CES.

In this paper, we extend the findings of these papers in the following way. First, we show that, under certain assumptions, the reasoning developed in Jones (2005) and Growiec (in press) can be reversed: one can also derive UFP distributions from firm-level production functions. Second, we apply our reasoning to Cobb–Douglas and CES functions, deriving their bilateral link to Pareto and Weibull distributions, respectively. In the course of our derivations, we stick to the Jones' (2005) assumption that marginal UFP distributions be independent. This assumption simplifies the subsequent analysis; relaxing it poses an interesting question for further research.

For simplicity, we also limit our setup to two production factors only (denoted K and L to make the reader think of capital and labor, respectively), but our results may be straightforwardly generalized to n factors of production. The proofs of propositions have been relegated to the Appendix A.

2. The transform

Our “endogenous technology choice transform” which links constant-returns-to-scale production functions with UFP distributions is valid under the following assumptions.

Assumption 1. Unit factor productivities of L and K , respectively \tilde{a} and \tilde{b} , are independent and their distributions have tail probabilities $P(\tilde{a} > a) = F_a(a)$ and $P(\tilde{b} > b) = F_b(b)$. Thus, the joint distribution of UFPs follows:

$$P(\tilde{a} > a, \tilde{b} > b) = F_a(a)F_b(b). \quad (1)$$

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Assumption 2. The technology frontier is a curve in the (a,b) space, such that the probability $P(\tilde{a} > a, \tilde{b} > b)$ is constant along this curve. At each given moment in time, the firm faces a unique technology frontier $F_a(a)F_b(b) = N$, where $N \in (0,1)$ is a constant.²

Assumption 3. Each particular production technique \tilde{Y} is given by a constant-returns-to-scale CES function

$$\tilde{Y}(K, L; a, b) = \tilde{A} \left(\psi (bK)^\theta + (1-\psi)(aL)^\theta \right)^{\frac{1}{\theta}}, \quad (2)$$

where $\tilde{A} > 0$, $\theta < 1$, $\theta \neq 0$, and $\psi \in (0,1)$. The most instructive case is the limit case $\theta \rightarrow -\infty$ which delivers a Leontief function. We denote $k \equiv K/L$. We also assume that θ is low enough to guarantee that the curvature of \tilde{Y} with respect to (a,b) is everywhere greater than the curvature of the technology frontier (cf. [Growiec, in press](#)).

Assumption 4. The firm operates in a competitive environment and chooses a technology (a,b) which is best in terms of attained profit. The set of available technologies is the technology frontier. Given Assumptions 1–3, this implies that for each (K,L) the firm seeks to:

$$\max_{a,b} \left\{ \tilde{A} \left(\psi (bK)^\theta + (1-\psi)(aL)^\theta \right)^{\frac{1}{\theta}} \right\} \text{ s.t. } F_a(a)F_b(b) = N. \quad (3)$$

Assumption 4 indicates the way the endogenous technology choice transform is going to be derived. The outcome of the firm's maximization problem – the production function $Y(K,L)$ – is going to be the convex hull of production techniques, whose shape depends upon the shape of the technology frontier. If one knows the shape of the technology frontier (tail probabilities F_a and F_b), then deriving the production function given CES production techniques is just a textbook application of the Lagrange multiplier method. Much less appreciated is the fact that the argument may be twisted around: knowing the shape of the production function Y , one may, in certain cases, derive the technology frontier and thus individual UFP distributions (given their independence). The following propositions hold.

Proposition 1. *The forward transform. Let the tail probabilities F_a, F_b be given. Denote their elasticities by: $\eta_a \equiv \frac{\partial F_a}{\partial a} \frac{a}{F_a}$ and $\eta_b \equiv \frac{\partial F_b}{\partial b} \frac{b}{F_b}$, respectively. The forward transform is effected in two steps:*

i. from the system of equations

$$\begin{cases} \frac{\psi}{1-\psi} \left(\frac{b}{a} \right)^\theta k^\theta = \frac{\eta_b(b)}{\eta_a(a)}, \\ F_a(a)F_b(b) = N, \end{cases} \quad (4)$$

derive $a(k)$ and $b(k)$;

ii. insert $a(k)$ and $b(k)$ into \tilde{Y} (defined as in Eq. (2)) to get the constant-returns-to-scale production function $Y(K,L)$.

Proposition 2. *The inverse transform. Let the production function Y be given. Denote its elasticities by: $\varepsilon_K \equiv \frac{\partial Y}{\partial K} \frac{K}{Y}$ and $\varepsilon_L \equiv \frac{\partial Y}{\partial L} \frac{L}{Y}$, respectively. By constant returns to scale, $\varepsilon_L = 1 - \varepsilon_K$. The inverse transform is effected in three steps:*

i. from the equation

$$\frac{\psi}{1-\psi} \left(\frac{b}{a} \right)^\theta k^\theta = \frac{\varepsilon_K(k)}{1-\varepsilon_K(k)}, \quad (5)$$

derive $k(b/a)$;

ii. insert $k(b/a)$ into the equation

$$\frac{\psi}{1-\psi} \left(\frac{b}{a} \right)^\theta k^\theta = \frac{\eta_b(b)}{\eta_a(a)} \quad (6)$$

to get first-order ordinary differential equations for F_a and F_b ;

iii. solve these ODEs to get the shapes of the UFP distributions.

The inverse transform seems to be a powerful analytical tool. Unfortunately, its applicability is limited, as is apparent from the following proposition:

Proposition 3. *The inverse transform can be effected only if the production function Y belongs to the CES family.*

The facts behind this finding are that (i) the constant-returns-to-scale property of particular production techniques with respect to (a,b) is necessarily inherited by the derived technology frontier³; at the same time, (ii) the inverse of the optimal technology choice function, $k(b/a)$ derived from Eq. (5), must be factorizable into functions of a only and b only because otherwise, the assumption of independence of UFP distributions would be violated. Consequently, as we show formally in the Appendix A, these two restrictions taken together imply that multiple functional forms must be ruled out of the analysis, and only for CES production functions will both conditions be satisfied simultaneously.

One possible way to get rid of this unwelcome feature of the inverse transform is to relax the assumption that the production techniques follow CES functions. We leave this for further work.

Apart from this negative result, we also get some interesting positive ones: Cobb–Douglas implies Pareto and vice versa, while CES implies Weibull and vice versa. These results may be obtained by applying the two transforms specified above. The following propositions hold.

Proposition 4. *The Cobb–Douglas production function is associated with Pareto distributions of UFPs:*

$$Y(K, L) = AK^\alpha L^{1-\alpha} \iff \begin{cases} F_a(a) = c_a a^\gamma \\ F_b(b) = c_b b^{\frac{\gamma\alpha}{1-\alpha}} \end{cases}$$

The ratio of exponents (shape parameters) of Pareto distributions is thus equal to $\frac{\alpha}{1-\alpha}$, but γ itself can be arbitrary. The parameters $A, c_a, c_b > 0$ and $\gamma < 0$ are also required to satisfy an additional equality restriction (see [Appendix A](#)).

Proposition 5. *The CES production function is associated with Weibull distributions of UFPs, with the same exponent α for F_a and F_b . The exponent α is related to the exponent ξ of the CES production function⁴ and the exponent θ of individual production techniques via the equality $\alpha = \frac{\xi\theta}{\xi-\theta}$, which reduces to $\alpha = -\xi$ for the case $\theta \rightarrow -\infty$ of Leontief production techniques.*

For some arbitrary $\zeta \in (0,1)$ and $\xi \in (-\infty, 0) \cup (0,1)$ satisfying the condition that either $\theta < \xi < 0$ or $0 < \theta < \xi$, we have:

$$Y(K, L) = A(\zeta K^\xi + (1-\zeta)L^\xi)^{1/\xi} \iff \begin{cases} F_a(a) = \exp\left(-\frac{c_a(\zeta-\theta)}{\xi\theta} a^{\frac{\xi\theta}{\xi-\theta}}\right) \\ F_b(b) = \exp\left(-\frac{c_b(\zeta-\theta)}{\xi\theta} b^{\frac{\xi\theta}{\xi-\theta}}\right) \end{cases}$$

³ There exists a range of cases for which the forward transform is applicable while the inverse transform is not. In such cases, the technology frontier does not have the constant-returns-to-scale property.

⁴ The constant elasticity of substitution σ is related to ξ via $\sigma = 1/(1-\xi)$.

² This “assumption” can actually be derived from a model of research and presented as a proposition. We refer to [Growiec \(in press\)](#) for such a derivation.

where $c_a, c_b, \zeta, A > 0$ satisfy two additional equality restrictions (see Appendix A).

It must be emphasized that the exponents of Pareto distributions F_a and F_b need not be equal in order to yield the Cobb–Douglas, while the exponents of Weibull distributions *must* be equal to yield the CES. If they are not equal, the resultant production function belongs to the wider “Clayton–Pareto” family defined and described in Growiec (in press).

3. Endogenous technology choice vs. aggregation

The results obtained herein might resemble the ones presented by Houthakker (1955–56) and Levhari (1968). There is one crucial difference between the two approaches, though: while we compute the *firm-level* production function as a convex hull of particular production techniques, characterized by UFPs picked from a given technology frontier endogenously by the representative firm, in Houthakker and Levhari, the *economy-wide* production function is calculated by aggregating inputs and outputs in an economy where heterogenous firms are characterized by Leontief technologies and unequal (but fixed) efficiency levels, distributed according to some distribution φ .

The original result of Houthakker was that if φ is Pareto then the aggregate production function is Cobb–Douglas. Levhari (1968) found that the reverse implication also holds and went on to show what distribution φ could be associated with the CES production function. His finding was that, using our notation, φ must have the density (Levhari, 1968):

$$g(x) = \frac{\zeta^{-1/(\zeta(1-\zeta))}}{(1-\zeta)(1-\zeta)^{1/\zeta}} x^{-\frac{2-\zeta}{1-\zeta}} \left(x^{\zeta/(1-\zeta)} - 1 \right)^{-\frac{1+\zeta}{\zeta}} \quad (7)$$

This is *not* a Weibull distribution. The apparent difference in results follows from an important difference between the two modeling approaches. The function $g(x)$ in Eq. (7) is the density of firms which require exactly x units of K for a unit of L to produce a single unit of output (x is thus firm-specific and is *not* a UFP of any productive factor). In Levhari (1968), firms cannot choose their preferred technologies as they can in the current paper; they are only allowed to stop producing (at no cost) if their profits become negative.

Mathematically, the difference is that our “aggregate” production function is a convex hull while Levhari’s (and Houthakker’s) is a definite integral.

Appendix A. Mathematical appendix

Proof of Proposition 1. From the FOCs of Eq. (3) with respect to a and b , we obtain

$$\tilde{A}^\theta \tilde{Y}^{1-\theta} \psi b^{\theta-1} K^\theta = \lambda F'_a F_a, \quad (A.1)$$

$$\tilde{A}^\theta \tilde{Y}^{1-\theta} (1-\psi) a^{\theta-1} L^\theta = \lambda F'_b F_b. \quad (A.2)$$

Dividing sidewise to get rid of the Lagrange multiplier λ , inserting appropriate formulas for F_a and F_b , and rewriting the technology frontier gives the system (4) in (i). Second order conditions of this problem are guaranteed to hold given Assumption 3 (this has been discussed in Growiec, in press). Assuming that a solution to the system (4) exists, we can insert $a(k)$ and $b(k)$ into Y to get $Y(K,L) = Ly(k)$ as the production function (being the convex hull of particular production techniques) in (ii). □

Proof of Proposition 2. The marginal productivities with respect to K and L of the production function and of particular production

techniques must be equal since the production function is the convex hull of production techniques. We have:

$$\tilde{A}^\theta \tilde{Y}^{1-\theta} \psi b^\theta K^{\theta-1} = \lambda \frac{\partial Y}{\partial K}, \quad (A.3)$$

$$\tilde{A}^\theta \tilde{Y}^{1-\theta} (1-\psi) a^\theta L^{\theta-1} = \lambda \frac{\partial Y}{\partial L}. \quad (A.4)$$

Dividing sidewise and rearranging yields Eq. (5) in (i). This is an equation in two variables, k and b/a . For the values for which the implicit function $k(b/a)$ exists, it can be inserted into Eq. (6), derived just above. Assuming that the left-hand side of Eq. (6) can be factorized into functions of a only and b only, we get two first-order ordinary differential equations: one for $F_a(a)$ and one for $F_b(b)$, as stated in (ii). Solving them and rearranging in a way that they describe tail probabilities of random variables gives (iii). □

Proof of Proposition 3. From Eq. (6), it follows that if one wants to carry out step (ii) of the inverse transform, $k(b/a)$ must be factorizable into functions of a only and b only. This means that $k(b/a) = g(a)h(b)$ for all a, b and some functions g, h . Differentiating both sides of this equation with respect to a and b yields:

$$k' \left(\frac{b}{a} \right) \frac{1}{a} = g'(a)h'(b), \quad (A.5)$$

$$-k' \left(\frac{b}{a} \right) \frac{b}{a^2} = g'(a)h(b). \quad (A.6)$$

Putting this together and rearranging, we get that

$$\frac{bh'(b)}{h(b)} = -\frac{ag'(a)}{g(a)}, \quad \forall a, b. \quad (A.7)$$

Since the left-hand side is a function of b only, and the right-hand side is a function of a only, Eq. (A.7) can be satisfied only if both expressions are constant. This implies that $k(b/a) = c(b/a)^\gamma$ for some constants c and γ . Inserting this into Eq. (5) and using the theorem due to Arrow et al. (1961) implies that the production function must be CES. □

Proof of Proposition 4. \Rightarrow : proofs in Jones (2005) and Growiec (in press). \Leftarrow : assume $Y(K,L) = AK^\alpha L^{1-\alpha}$, with $\alpha \in (0,1)$. From Eq. (5), we have

$$\frac{\psi}{1-\psi} \left(\frac{b}{a} \right)^\theta k^\theta = \frac{\alpha}{1-\alpha}. \quad (A.8)$$

For second order conditions of optimization in Eq. (3) to be satisfied, it must be the case that $\theta < 0$ (Growiec, in press). It follows that $\frac{\eta_b}{\eta_a} = \frac{\alpha}{1-\alpha}$ so η_b and η_a are constant. This means that $F_a(a) = c_a a^\gamma$ and $F_b(b) = c_b b^{\gamma-\alpha}$ for some arbitrary $\gamma < 0$ and $c_a, c_b > 0$ related to $A > 0$ via the equality:

$$A = \tilde{A} \left(\frac{N}{c_a c_b} \right)^{\frac{1-\alpha}{\gamma}} \left(\psi^\alpha (1-\psi)^{1-\alpha} \left(\left(\frac{\alpha}{1-\alpha} \right)^{1-\alpha} + \left(\frac{1-\alpha}{\alpha} \right)^\alpha \right) \right)^{\frac{1}{\gamma}}, \quad (A.9)$$

which simplifies to $A = \tilde{A} \left(\frac{N}{c_a c_b} \right)^{\frac{1-\alpha}{\gamma}}$ in the limit case of $\theta \rightarrow -\infty$. Hence, UFPs are Pareto-distributed. Eq. (A.9) can be easily derived by applying the forward transform (Growiec, in press). □

Proof of Proposition 5. \Rightarrow : The technology frontier is given by $e^{-\gamma a^\alpha - b^\alpha} = \tilde{N}$ which is easily transformed to $\gamma a^\alpha + b^\alpha = \tilde{N}$. The proof for this case uses the theorem due to Arrow et al. (1961) and can be found in Growiec (in press).

\Leftarrow : assume $Y(K,L) = A (\zeta K^\xi + (1-\zeta)L^\xi)^{1/\xi}$, with $\zeta \in (0,1)$ and $\xi < 1, \xi \neq 0$. For second order conditions of optimization in Eq. (3) to be satisfied, it

must be the case that $\theta < \xi$ (Growiec, in press). From Eq. (5), we have $\frac{\psi}{1-\psi} \left(\frac{b}{a}\right)^\theta k^\theta = \frac{\zeta}{1-\zeta} k^\xi$. Rearranging this yields: $k(b/a) = \left[\frac{\psi}{1-\psi} \frac{1-\zeta}{\zeta} \left(\frac{b}{a}\right)^\theta\right]^{\frac{1}{\xi-\theta}}$. Denoting $c = \left(\frac{\psi}{1-\psi}\right)^{\frac{\xi}{\xi-\theta}} \left(\frac{1-\zeta}{\zeta}\right)^{\frac{\theta}{\xi-\theta}}$, we get in step (ii) that $\frac{\eta_b(b)}{\eta_a(a)} = c \left(\frac{b}{a}\right)^{\frac{\xi\theta}{\xi-\theta}}$. Let us assume $c_a, c_b > 0$ to be some arbitrary constants satisfying $c_b = c_a c$. Solving the resultant differential equations $F_b'(b)b = -F_b(b)c_b b^{\frac{\xi\theta}{\xi-\theta}}$, $F_a'(a)a = -F_a(a)c_a a^{\frac{\xi\theta}{\xi-\theta}}$ and imposing that F_a, F_b be equal to 1 at 0 yields:

$$F_a(a) = \exp\left(\frac{-c_a(\zeta-\theta)}{\zeta\theta} a^{\frac{\xi\theta}{\xi-\theta}}\right), \quad (\text{A.10})$$

$$F_b(b) = \exp\left(\frac{-c_b(\zeta-\theta)}{\zeta\theta} b^{\frac{\xi\theta}{\xi-\theta}}\right), \quad (\text{A.11})$$

Thus, UPFs are Weibull-distributed. Apart from $c_b = c_a c$, the following equality restriction must also hold, relating c_b, ζ and A :

$$A = \tilde{A} \left(\frac{-\ln N}{c_b}\right)^{\frac{\xi-\theta}{\zeta\theta}} \psi^{\frac{1}{\theta}} \zeta^{-1}. \quad (\text{A.12})$$

Eq. (A.12) simplifies to $A = \tilde{A} \left(\frac{-\ln N}{c_b} \cdot \zeta\right)^{-\frac{1}{\zeta}}$ in the limit case of $\theta \rightarrow -\infty$ and it can be easily derived by applying the forward transform (Growiec, in press).

It must also be noted that in the case where $\xi > 0$ and $\theta < 0$, second order conditions hold but F_a and F_b cannot describe tail probabilities: $\lim_{a \rightarrow \infty} F_a(a) > 0$, $\lim_{b \rightarrow \infty} F_b(b) > 0$. Hence, such case must also be ruled out from our considerations. \square

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