Static Models of Oligopoly
Cournot and Bertrand Models

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Outline

1. Introduction
   - Game Theory and Oligopolies

2. The Bertrand Model
   - Basic Model
   - N firms model
   - Diversified product

3. The Cournot Model
   - Basic Model
   - Numerical Example
   - N firms setting
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   - N firms setting
Oligopoly - definition

More than one firm, but not too many
Oligopoly - definition

- More than one firm, but not too many
- Similar products (or the same)
Oligopoly - definition

Frame subtitles are optional. Use upper- or lowercase letters.

- More than one firm, but not too many
- Similar products (or the same)
- Static oligopoly
Why Game Theory?
Frame subtitles are optional. Use upper- or lowercase letters.

- Non continuous profit functions
Why Game Theory?
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- Non continuous profit functions
- Leaders and Followers
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- Non continuous profit functions
- Leaders and Followers
- Cartels
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Basic Bertrand Model

Two Players (duopoly)
Basic Bertrand Model
Frame subtitles are optional. Use upper- or lowercase letters.

- Two Players (duopoly)
- Players choose price
Basic Bertrand Model
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- Two Players (duopoly)
- Players choose price
- Symetric production (profit) functions
Basic Bertrand Model

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- Two Players (duopoly)
- Players choose price
- Symmetric production (profit) functions
- Constant marginal cost
Basic Bertrand Model

- Two Players (duopoly)
- Players choose price
- Symmetric production (profit) functions
- Constant marginal cost
- Demand linear, decreasing with $p$: $x(p)$
Mathematical representation

- Demand function
Mathematical representation

\[ x_j(p_j, p_k) = \begin{cases} 
  x(p_j) & \text{if } p_j < p_k \\
  \frac{1}{2}x(p_j) & \text{if } p_j = p_k \\
  0 & \text{if } p_j > p_k 
\end{cases} \quad (1) \]

- Demand function
Mathematical representation

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- Demand function
- Profit function
Mathematical representation

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  \frac{1}{2}x(p_j) & \text{if } p_j = p_k \\
  0 & \text{if } p_j > p_k
\end{cases}
\]  (1)

- Demand function

\[
\pi_j(p_j, p_k) = x_j(p_j, p_k)(p_j - c)
\]  (2)

- Profit function
Solution

Proposition 1

There is an unique Nash equilibrium \((p_j^*, p_k^*)\) in the Bertrand duopoly model. In this equilibrium, both firms set their prices equal to cost: \(p_j^* = p_k^* = c\).
Solution

Proposition 1

There is a unique Nash equilibrium \((p^*_j, p^*_k)\) in the Bertrand duopoly model. In this equilibrium, both firms set their prices equal to cost: \(p^*_j = p^*_k = c\).

Proof
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   - \( N \) firms model
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   - \( N \) firms setting
Bertrand with N players

- Firms problems are exactly the same
Bertrand with $N$ players

- Firms problems are exactly the same
- This time $N$ players compete ($N > 2$)
Bertrand with N players

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- This time N players compete ($N > 2$)

**Corollary**

*In setting with $N > 2$ firms Bertrand model of oligopoly produces exactly same results as with $N = 2$ firms*
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Bertrand with diversified product

- Two Players.
- Products are not perfect subsidies
- Example: spatial model
  - Reservation Price $V > c$
  - $t$ - cost of 'traveling'
  - $N$ - number of customers
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Basic Model

- Two Players (duopoly)
Basic Model

- Two Players (duopoly)
- Players chose quantity
Basic Model

- Two Players (duopoly)
- *Players chose quantity*
- Symmetric profit functions
Basic Model

- Two Players (duopoly)
- Players chose quantity
- Symmetric profit functions
- Constant marginal cost
Basic Model

- Two Players (duopoly)
- Players chose quantity
- Symmetric profit functions
- Constant marginal cost
- Inverse demand function $p(q)$ linear, decreasing with
  $q = \sum_{j=1}^{N} q_j$
Players’ Problems

- Maximization problem

\[ \text{j-th player faces problem:} \]
\[ \max_{q_j \geq 0} p(q_j + \bar{q}_k)q_j - cq_j \]  (3)
Players’ Problems

- Maximization problem

\[ \text{Max} \quad p(q_j + \bar{q}_k)q_j - cq_j \quad \text{for} \quad q_j \geq 0 \quad (3) \]

- In this setting, firm j faces inverse demand function, like monopolist: \( \tilde{p}(q_j + \bar{q}_k) \)
Players’ Problems

- Maximization problem

\[ \text{Max}_{q_j \geq 0} p(q_j + q_k)q_j - c q_j \]  

\( j \)-th player faces problem:

- in this setting firm \( j \) faces inverse demand function, like monopolist: \( \tilde{p}(q_j + q_k) \)

- Solution

\[ p'(q_j + q_k)q_j + p(q_j + q_k) \leq c \quad \text{with equality if } q_j > 0 \]
Players’ Problems

- Maximization problem

\[ \text{Max}_{q_j \geq 0} p(q_j + \bar{q}_k)q_j - cq_j \]  

- in this setting firm \( j \) faces inverse demand function, like monopolist: \( \tilde{p}(q_j + \bar{q}_k) \)

- Solution

\[ p'(q_j + \bar{q}_k)q_j + p(q_j + \bar{q}_k) \leq c \quad \text{with equality if } q_j > 0 \]  

- Let \( b_j(\bar{q}_k) \) denote set of optimal responses of player \( j \), given strategy (quantity) of player \( k \)
Nash Equilibrium

NE if and only if:

\[
p' (q_j^* + q_k^*) q_j^* + p (q_j^* + q_k^*) \leq c \tag{5}
\]

and

\[
p' (q_j^* + q_k^*) q_k^* + p (q_j^* + q_k^*) \leq c \tag{6}
\]
Nash Equilibrium

- NE if and only if:

\[ p' (q_j^* + q_k^*) q_j^* + p (q_j^* + q_k^*) \leq c \]  \hspace{1cm} (5)

and

\[ p' (q_j^* + q_k^*) q_k^* + p (q_j^* + q_k^*) \leq c \]  \hspace{1cm} (6)

Let's add (5) and (6):

\[ p' (q_j^* + q_k^*) \frac{(q_j^* + q_k^*)}{2} + p (q_j^* + q_k^*) = c \]  \hspace{1cm} (7)

**Proposition 2**

In any Nash equilibria of the Cournot duopoly model with costs \( c > 0 \) per unit for the two firms and an inverse demand function \( p(\cdot) \) satisfying \( p'(q) < 0 \) for all \( q \geq 0 \) and \( p(0) > c \), the market price is greater than \( c \) (the competitive and smaller than monopoly price).
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Let $q = \sum_{j=1}^{N} q_j$ and $N = 2$, inverse demand is given by:

$$p(q) = a - bq$$

Cost functions are linear, that is $c_j(q_j) = c q_j$
Problem

Let \( q = \sum_{j=1}^{N} q_j \) and \( N = 2 \), inverse demand is given by:

\[
p(q) = a - bq
\]  
(8)

Cost functions are linear, that is \( c_j(q_j) = cq_j \)

- **Solution**
  - solve for monopolist \( (N = 1) \) (find \( q^m \));
Let \( q = \sum_{j=1}^{N} q_j \) and \( N = 2 \), inverse demand is given by:

\[
p(q) = a - bq
\]  

(8)

Cost functions are linear, that is \( c_j(q_j) = cq_j \)

- **Solution**
  - solve for monopolist \( (N = 1) \) (find \( q^m \));
  - solve for competitive markets \( (p^\circ = c) \) (find \( q^\circ \));
Introduction

The Bertrand Model

The Cournot Model

Summary

Problem

Let \( q = \sum_{j=1}^{N} q_j \) and \( N = 2 \), inverse demand is given by:

\[
p(q) = a - bq
\]

(8)

Cost functions are linear, that is \( c_j(q_j) = cq_j \)

Solution

- solve for monopolist \( (N = 1) \) (find \( q^m \));
- solve for competitive markets \( (p^\circ = c) \) (find \( q^\circ \));
- find best response functions for each firm;
Problem

Let \( q = \sum_{j=1}^{N} q_j \) and \( N = 2 \), inverse demand is given by:

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p(q) = a - bq
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(8)

Cost functions are linear, that is \( c_j(q_j) = cq_j \)

- Solution
  - solve for monopolist (\( N = 1 \)) (find \( q^m \));
  - solve for competitive markets (\( p^o = c \)) (find \( q^o \));
  - find best response functions for each firm;
  - ...

Problem

Let \( q = \sum_{j=1}^{N} q_j \) and \( N = 2 \), inverse demand is given by:

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  - solve for monopolist \( (N = 1) \) (find \( q^m \));
  - solve for competitive markets \( (p^o = c) \) (find \( q^o \));
  - find best response functions for each firm;
  - ...
  - profit!
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Cournot with N players

- Firms face exact same problems
- N firms compete
Cournot with N players

- Firms face exact same problems
- N firms compete
- Equation (7) takes following form with N firms:

\[ p'(q^*) \frac{(q^*)}{N} + p(q^*) = c \], where \( q^* \) is aggregate output in NE  \( (9) \)
Cournot with N players

- Firms face exact same problems
- N firms compete
- Equation (7) takes following form with N firms:

\[ p' \left( q^* \right) \frac{\left( q^* \right)}{N} + p( q^* ) = c \] , where \( q^* \) is aggregate output in NE

(9)

- if \( N \to \infty \) equation (9) takes form:

\[ p( q^* ) = c = p^\circ \] 

(10)
Cournot with N players

- Firms face exact same problems
- N firms compete
- Equation (7) takes following form with N firms:

\[ p'(q^*) \left( \frac{q^*}{N} \right) + p(q^*) = c \]  
where \( q^* \) is aggregate output in NE

- if \( N \to \infty \) equation (9) takes form:

\[ p(q^*) = c = p^\circ \]  

- if \( N = 1 \) equation (9) takes form:

\[ p'(q^*) q^* + p(q^*) = c \]
Cournot with $N$ players

- Firms face exact same problems
- $N$ firms compete
- Equation (7) takes following form with $N$ firms:

$$p'(q^*) \frac{(q^*)}{N} + p(q^*) = c,$$
where $q^*$ is aggregate output in NE

(9)

- if $N \to \infty$ equation (9) takes form:

$$p(q^*) = c = p^\circ$$

(10)

- if $N = 1$ equation (9) takes form:

$$p'(q^*) q^* + p(q^*) = c$$

(11)

which is monopolist solution
### Summary

- The Bertrand Model
- The Cournot Model
- Strategic Substitutes vs Strategic Complements
- Outlook
  - Dynamic Games